Turbulent energy cascade:
a Non-Equilibrium Statistical Mechanics perspective

Niccolò Cocciaglia

joint work with M. Cencini and A. Vulpiani

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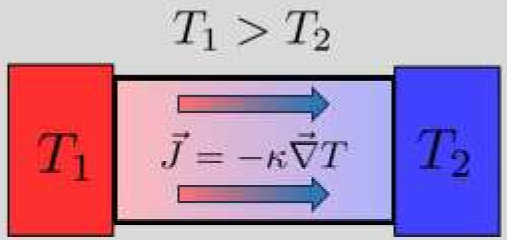
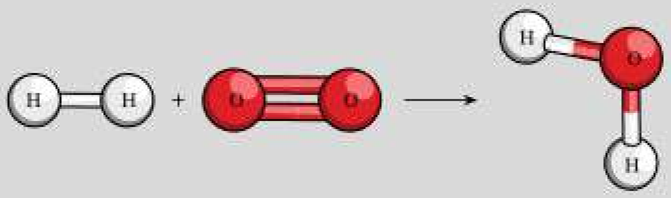
SMART-TURB seminar – 24/10/2023

Out of equilibrium processes



Physics



Chemistry

Biology

Common features:

- Preferential direction of ''currents'' caused by ''forces''
- Breaking of time-reversal symmetry
- Positive entropy production

$\Downarrow$

Some kind of ''asymmetry'' is always present

Turbulence and energy cascades

Navier-Stokes equation:

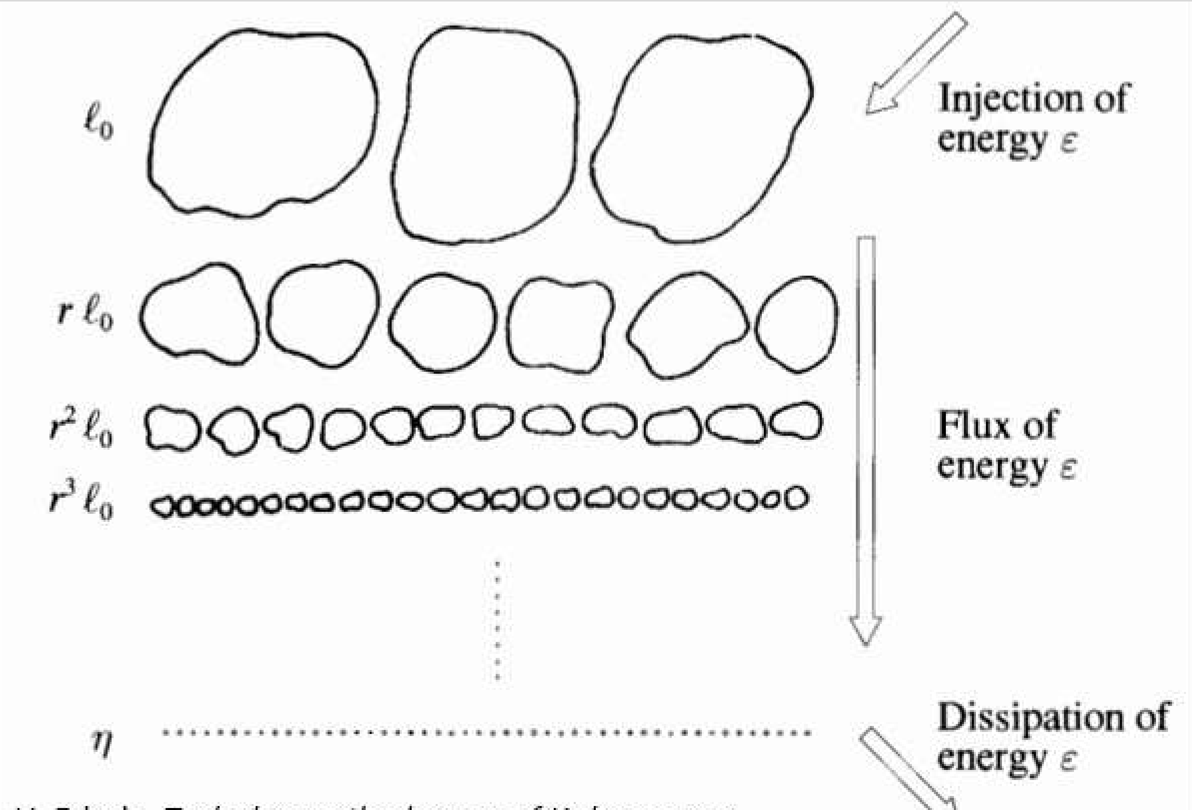
$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \mathbf{f} \\ \nabla\cdot\mathbf{v} = 0 \end{cases}$$

advection, pressure, dissipation, forcing

Reynolds number: $Re = \frac{VL}{\nu} \sim \frac{inertia}{dissipation}$

$Re \sim O(10^3)$

$Re \sim O(10^7)$



U. Frisch, *Turbulence: the legacy of Kolmogorov*

Energy balance (stationarity):

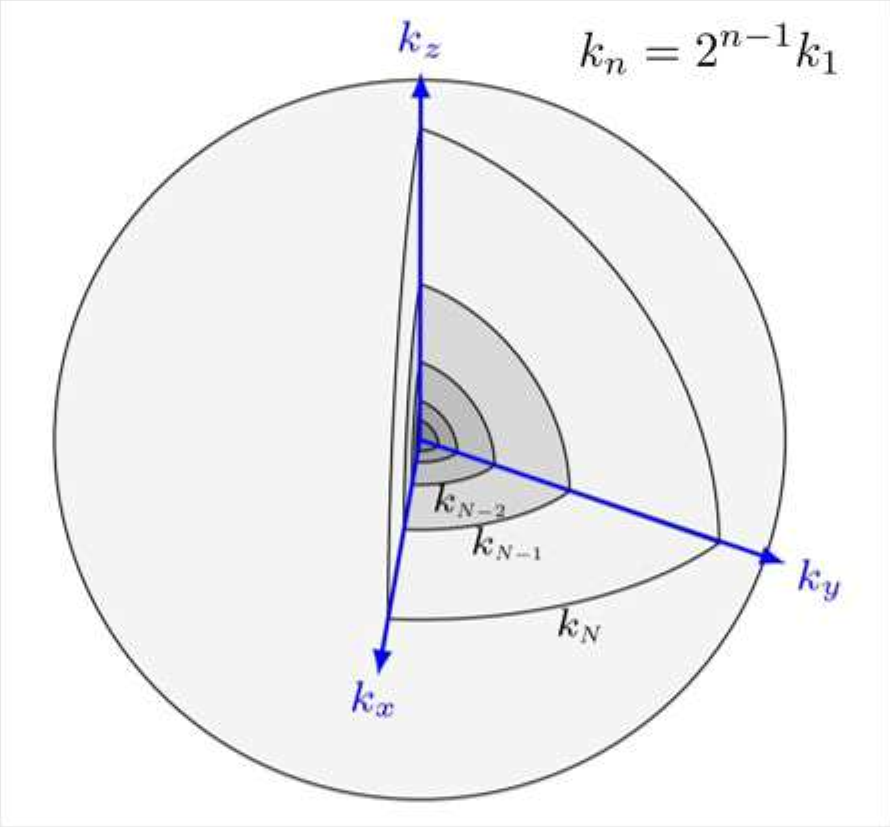
$$\langle injection\ rate\rangle = \langle flux\rangle = \langle dissipation\ rate\rangle = \epsilon$$

Shell models

Navier-Stokes, Fourier space:

$$\left(\partial_t + \nu k^2\right) \widehat{u}_i(\mathbf{k}) = -ik_m \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \sum_{\substack{\mathbf{k}_1, \mathbf{k}_2 \\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}} \widehat{u}_j(\mathbf{k}_1)\, \widehat{u}_m(\mathbf{k}_2) + \widehat{f}_i$$

Triad interactions



Recipe to create your shell model:

1. Discretize Fourier space in spherical shell with exponentially-growing radii
2. Retain only one scalar complex variable per shell:

$$u_n \in \mathbb{C}\,, \quad n = 1, \ldots, N$$

3. Only local interactions, but keeping same conservation laws for $\nu = f_n = 0$

BEST RESULT:

SABRA shell model (L'vov et al., PRE 1998)

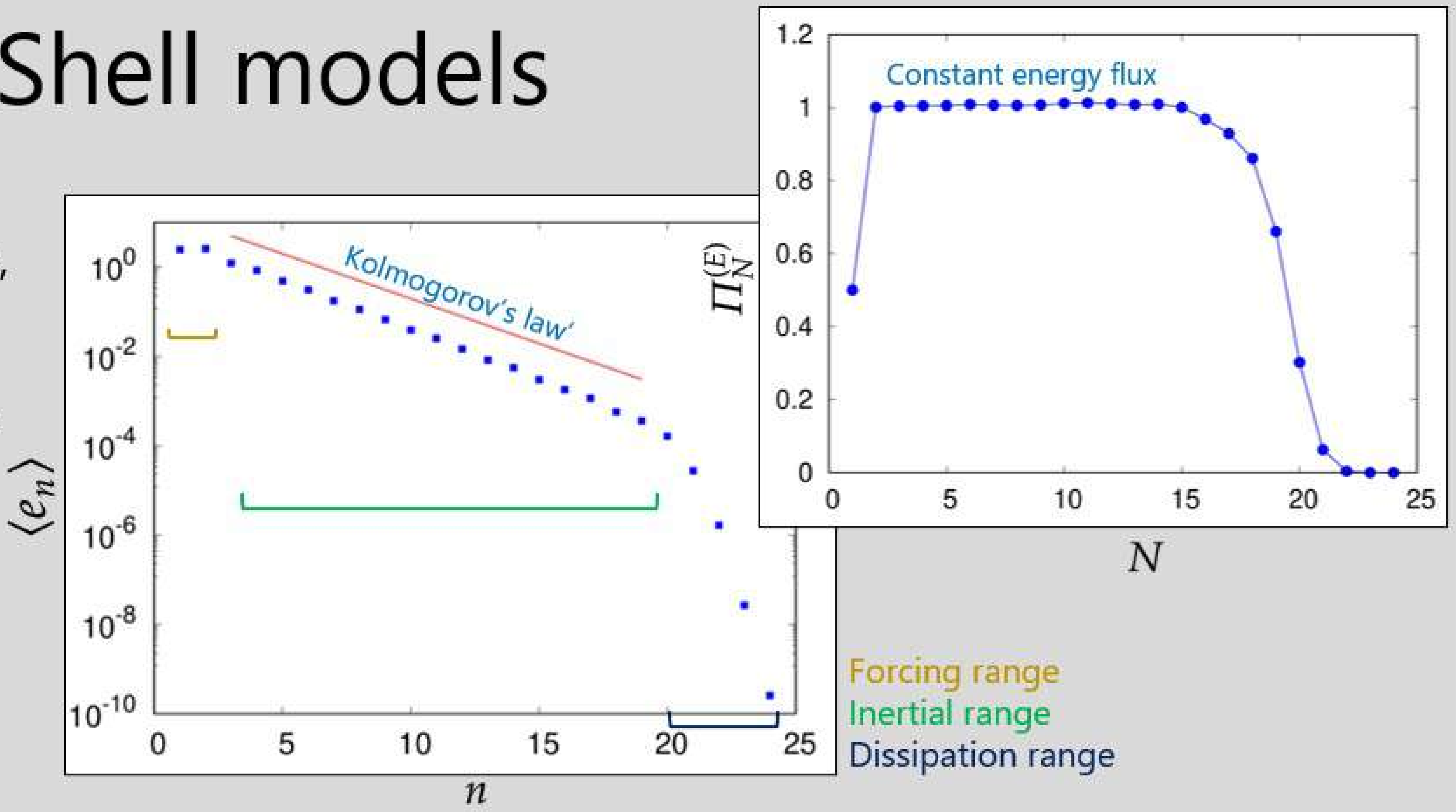
$$\left(\frac{d}{dt} + \nu k^2\right) u_n = i \left[k_{n+1}\, u^*_{n+1}\, u_{n+2} + b\, k_n\, u^*_{n-1}\, u_{n+1} + (1+b)\, k_{n-1}\, u_{n-2}\, u_{n-1}\right] + f_n$$

If $\nu = f_n = 0$: similar to truncated Euler equation $\Longrightarrow$ Equilibrium

Shell models

PROs

- Fast numerical computations (few DOFs, set of ODEs)
- Good estimation of statistical quantities
- Most turbulent features are preserved (intermittency, anomalous scaling, Kolmogorov's 4/5-law)



CONs

- Vectorial properties (geometry) are lost

(Moin and Mahesh, Annu. Rev. Fluid Mech., 1998)

Irreversibility: how to measure it

ENTROPY PRODUCTION

$$\Delta S(t) = k_B \ln \frac{P(\{x(s)\}_{0<s<t})}{P(\{\mathcal{I}x(s)\}_{0<s<t})}$$

(Lebowitz, Spohn, J. Stat Phys., 1999)

DOWNSIDES
- Hard to compute
- Global quantity

TIME CORRELATION FUNCTIONS

Systems at equilibrium: $\langle A(t)B(t+\tau)\rangle = \langle B(t)A(t+\tau)\rangle$ (microscopic reversibility)
(Onsager, Phys. Rev. 1931)

Systems out of equilibrium: $\langle A(t)B(t+\tau)\rangle \neq \langle B(t)A(t+\tau)\rangle$ ← Here, a careful choice of A and B has to be made!
(Pomeau, J. Phys. France 1982)

Must be 'signals' which are not symmetric under time reversal

$$\Downarrow$$

$$\Psi(\tau) = \langle A(t)B(t+\tau) - B(t)A(t+\tau)\rangle \neq 0$$

Anti-symmetric time correlation function

Cascades: a source of irreversibility

Energy flux has a preferential direction: from forcing scale to dissipation scale

AIM:
Seek connection between energy cascade and asymmetry under time reversal

TOOL:
Anti-symmetric correlation functions just introduced

Our pick:

$$\Psi_{e_n}(\tau) = \frac{\langle e_n^2(t) e_n(t+\tau)\rangle - \langle e_n(t) e_n^2(t+\tau)\rangle}{\langle e_n^3(t)\rangle} \quad \text{where} \quad e_n(t) = \frac{1}{2}|u_n(t)|^2$$

ADVANTAGES
- Easy to compute
- 'Local' quantity

A large variety of odd functions works as well, but this one allows for a physical interpretation...

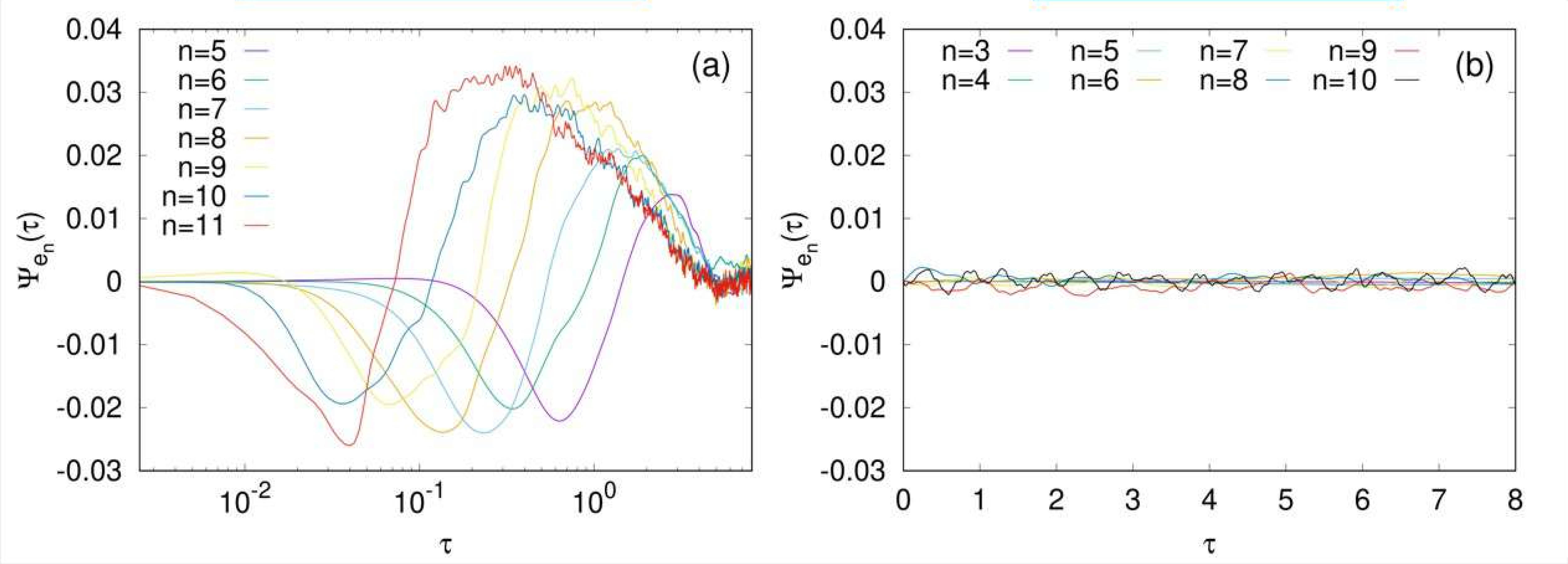
The same functional form was adopted in:
Josserand et. al., J. Stat. Phys. 2017

Cascades: a source of irreversibility

$$\Psi_{e_n}(\tau) = \frac{\langle e_n^2(t) e_n(t+\tau)\rangle - \langle e_n(t) e_n^2(t+\tau)\rangle}{\langle e_n^3(t)\rangle}$$

TURBULENT $\nu \neq 0,\ f_n \neq 0$

INVISCID $\nu = 0,\ f_n = 0$



Cascades: a source of irreversibility

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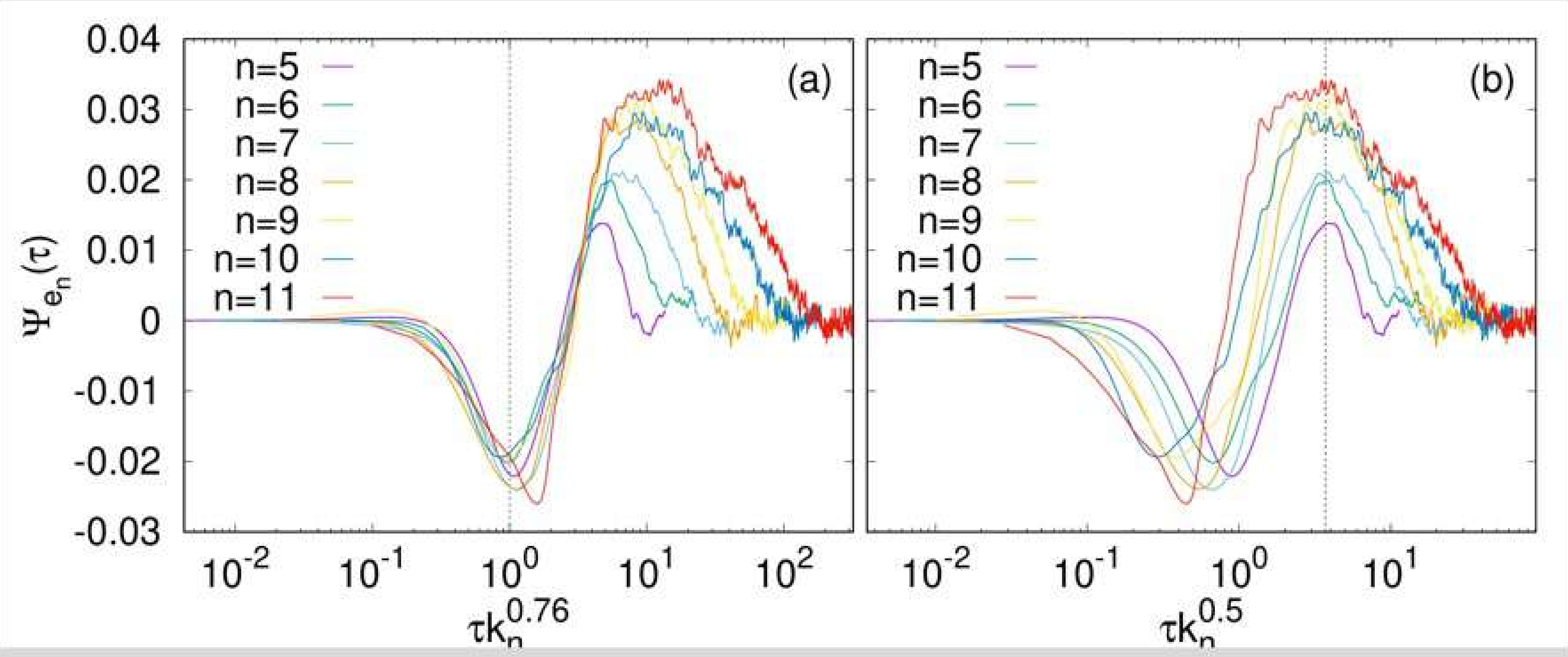


A unique time rescaling for aligning both minima and maxima cannot be found ⟶ Contribution from <u>many</u> scales (from slowest to that of shell n) ⟶ **Multi-time multi-scale correlation functions**

(Biferale et al., Physica D, 1999)

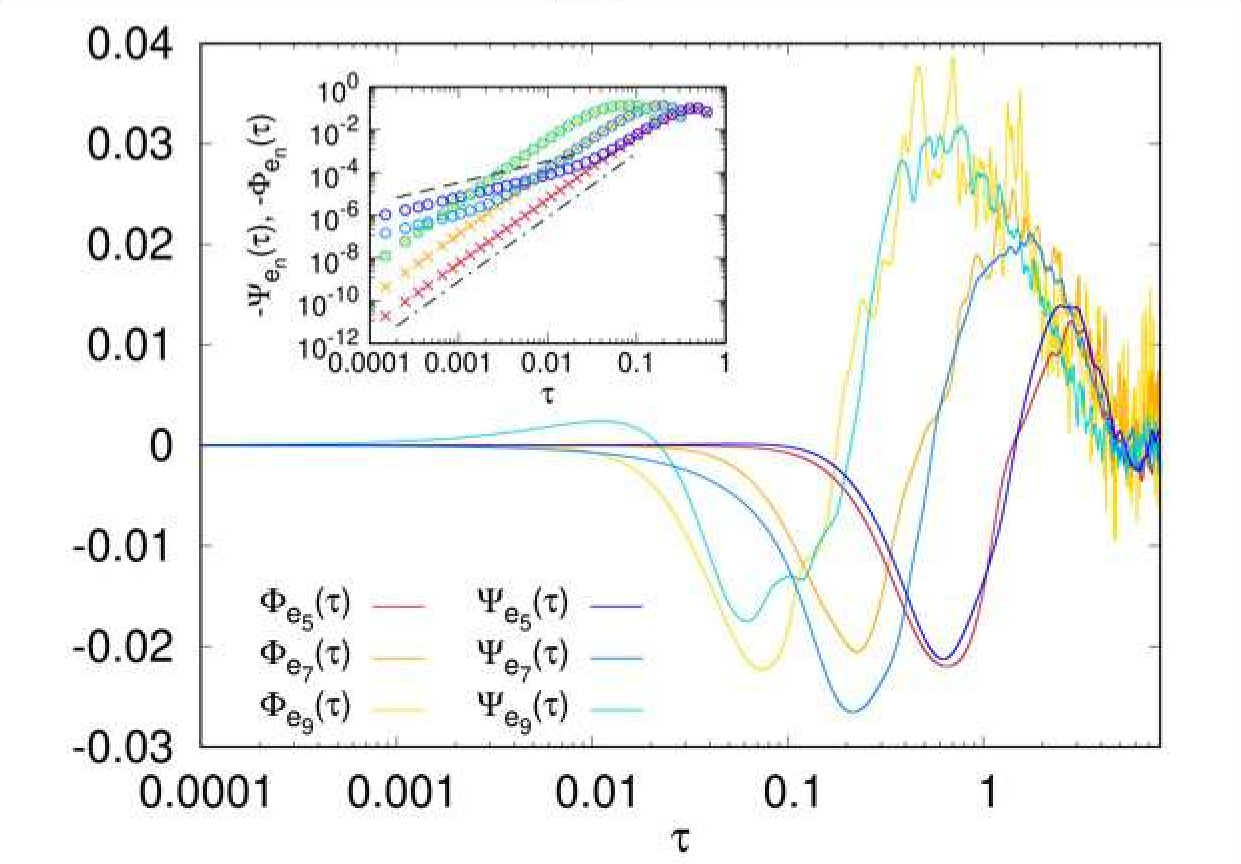
Physical interpretation

$$\Psi_{e_n}(\tau) = \frac{\langle e_n^2(t)e_n(t+\tau)\rangle - \langle e_n(t)e_n^2(t+\tau)\rangle}{\langle e_n^3(t)\rangle}$$

equivalent to
(under stationarity HP)

$$\Phi_{e_n}(\tau) = \frac{1}{3}\frac{\langle [e_n(t+\tau) - e_n(t)]^3\rangle}{\langle e_n^3(t)\rangle}$$

Measure of local energy gains and losses



Faster losses and slower gains, as expected from scaling laws

Small time separation (inset):

$$\Psi_{e_n}(\tau \simeq 0) = -\langle e_n^2 \dot{e}_n\rangle \tau + \mathcal{O}(\tau^2)$$

$$\Phi_{e_n}(\tau \simeq 0) = \frac{1}{3}\langle \dot{e}_n^3\rangle \tau^3 + \mathcal{O}(\tau^4)$$

Stationarity: should be 0, but linear dependence is found in simulations

Previous justification (Josserand et al): finite-time singularities

Our justification (supported by numerics): statistical convergence

Physical interpretation

$$\Psi_{e_n}(\tau) = \frac{\langle e_n^2(t) e_n(t+\tau)\rangle - \langle e_n(t) e_n^2(t+\tau)\rangle}{\langle e_n^3(t)\rangle}$$

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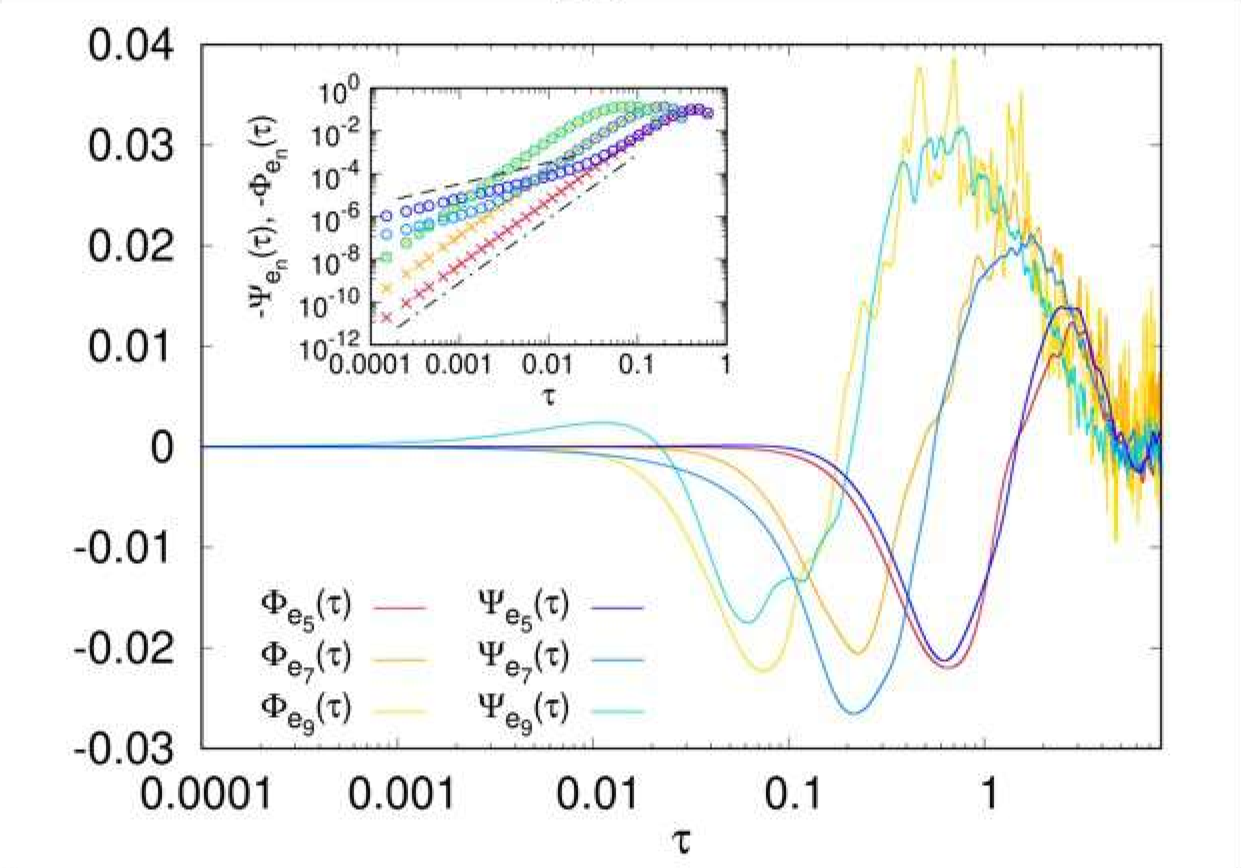

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Statistics of $\dot{e}_n$ determines initial-time behaviour

Previous studies on power statistics

Flight–crash events in turbulence

Haitao Xu[a,b], Alain Pumir[a,b,c], Gregory Falkovich[a,d,e], Eberhard Bodenschatz[a,b,f,g,1], Michael Shats[h], Hua Xia[h], Nicolas Francois[h], and Guido Boffetta[a,i]

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- Tracer fluid particle in turbulent flow
- Irreversibility in terms of power fluctuations of the tracer along a trajectory

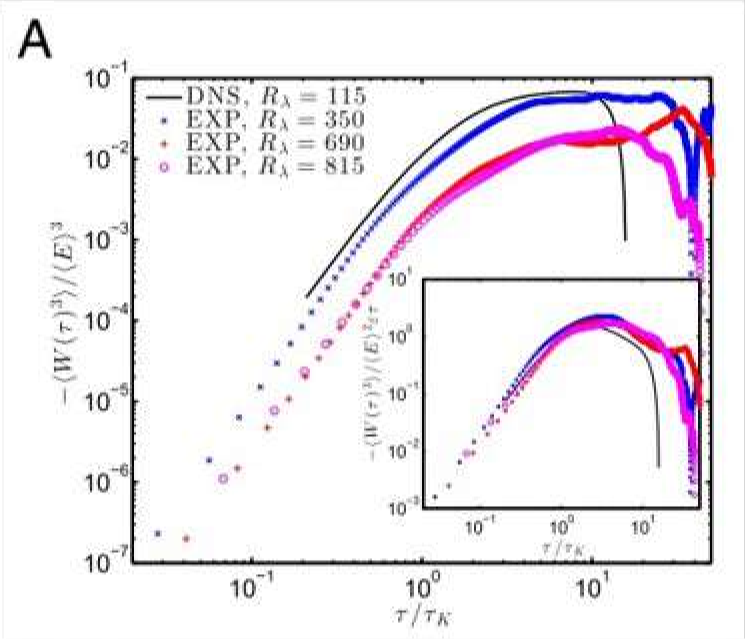
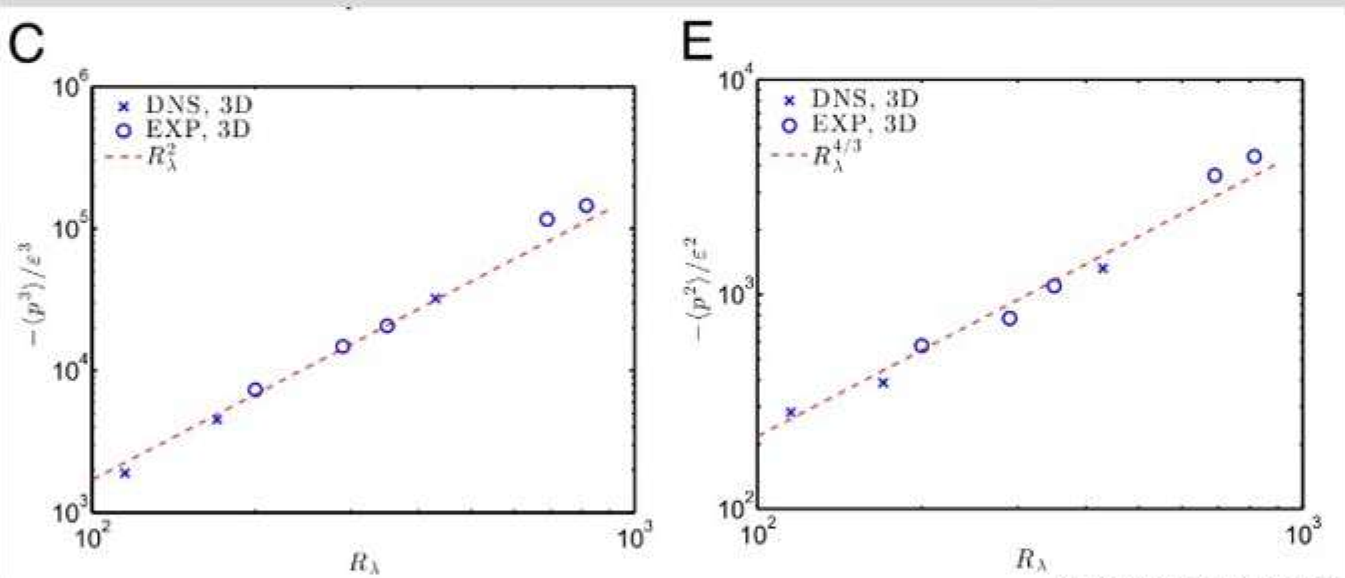


Fig. 2. (A) The third moment of energy increments $W(\tau) = E(t+\tau) - E(t)$ as a function of τ in 3D turbulence for different Reynolds numbers from both experiments (EXP) and direct numerical simulations (DNS). The quantity $-\langle W(\tau)^3 \rangle$ grows like τ^3 at short times. The curves obtained at different Reynolds numbers collapse once scaled using [3]

$$W(\tau) = E(t+\tau) - E(t)$$

$$p = \lim_{\tau \to 0} W(\tau)/\tau = dE/dt$$



(C–F) Statistical properties of the instantaneous power p acting on fluid particles. (C) Variation of $-\langle p^3 \rangle/\varepsilon^3$ vs. R_λ for 3D turbulence. Its increase is close to R_λ^2. (D) Variation of $-\langle p^3 \rangle/\varepsilon^3$ vs. R_α for 2D turbulence, which increases approximately as R_α^2. (E and F) Variation of $\langle p^2 \rangle/\varepsilon^2$ for 3D and 2D turbulence, respectively. The variance increases rapidly with Reynolds numbers, close to $R_\lambda^{4/3}$ or $R_\alpha^{4/3}$ for 3D and 2D turbulence. This results in a skewness nearly independent of the Reynolds number: $\langle p^3 \rangle / \langle p^2 \rangle^{3/2} \approx -0.5$ in 3D and ≈ -0.20 in 2D.

Previous studies on power statistics

PHYSICAL REVIEW FLUIDS **2**, 104604 (2017)

Time irreversibility and multifractality of power along single particle trajectories in turbulence

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(Received 27 July 2017; published 27 October 2017)

- DNS and shell model simulations
- Irreversibility in terms of symmetric and non-symmetric moments of the power

DNS

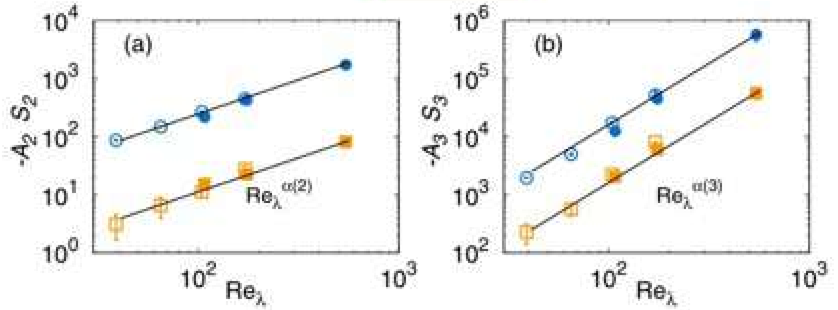


FIG. 2. Scaling behavior of Lagrangian power moments (9) $\mathcal{S}_q$ (blue circles) and $-\mathcal{A}_q$ (orange squares) for (a) $q = 2$ and (b) $q = 3$. Data refer to DNS1 (closed symbols) and DNS2 (open symbols) data sets, described in the Appendix. Solid lines show the slopes (a) $\alpha(2) = 1.17$ and (b) $\alpha(3) = 2.1$ predicted by the MF via (8) with (2) for $\beta = 0.6$. Errors bars have been obtained as standard errors over independent configurations of the turbulent field. We used from 5 to 40 configurations spaced by approximately T_L, depending on the resolution.

$$\mathcal{S}_q = \langle |p|^q \rangle / \epsilon^q$$

$$\mathcal{A}_q = \langle p|p|^{q-1} \rangle / \epsilon^q$$

SABRA

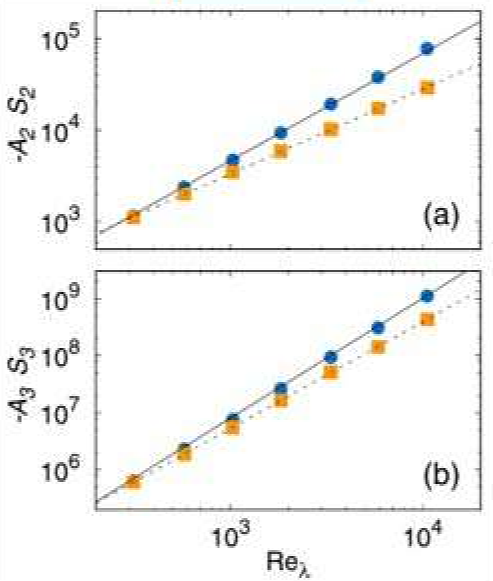
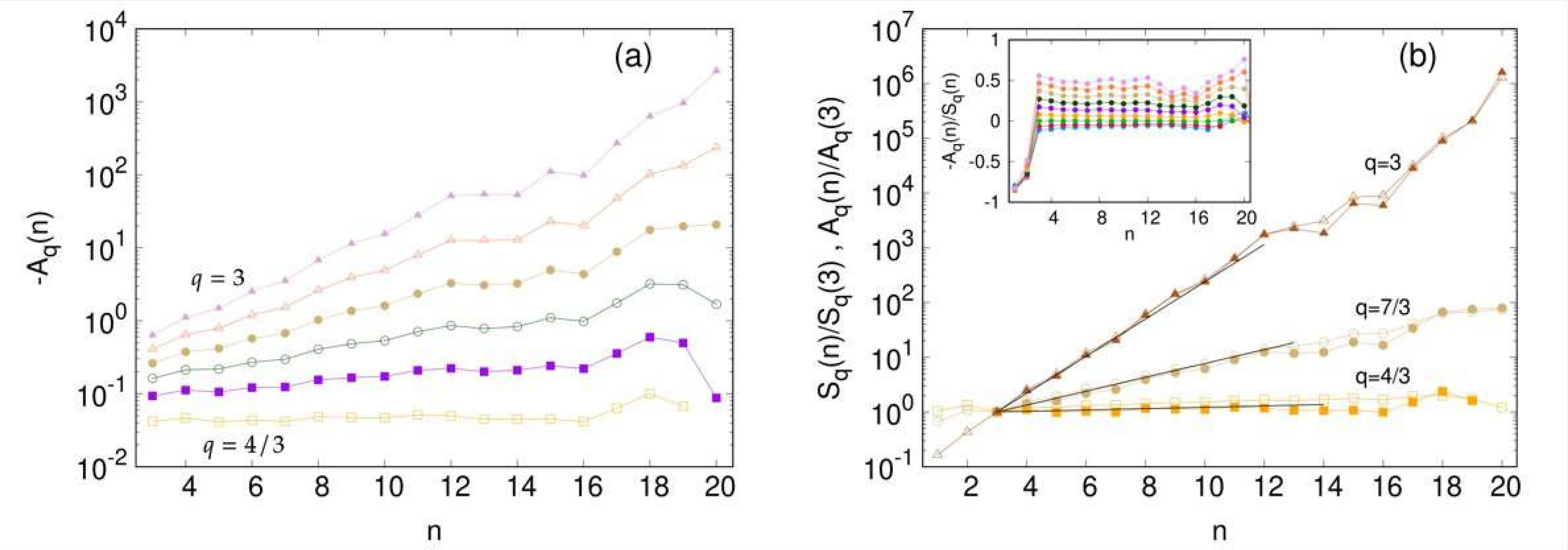


FIG. 4. Lagrangian power statistics in the shell model with $N = 30$ shells at varying ν. The Re_λ dependence of $\mathcal{S}_q$ and $-\mathcal{A}_q$ is shown for (a) $q = 2$ and (b) $q = 3$ compared with the MF prediction (8) (solid lines) and the best fit of the asymmetry-sensitive observables (dashed lines) providing slopes (a) 0.93(1) and (b) 1.87(1). Notice that $-\mathcal{A}_q$ is shifted upward to highlight the different scaling behavior.

Statistics of shell powers

Let us consider : $p_n = \dot{e}_n$ and its symmetric and non-symmetric moments

$$S_q(n) = \langle |p_n|^q \rangle / \epsilon^q$$
$$A_q(n) = \langle p_n |p_n|^{q-1} \rangle / \epsilon^q$$



$A_q(n) \neq 0 \implies$ Shell energies vary asymmetrically in time

Statistics of shell powers

Let us consider : $p_n = \dot{e}_n$ and its symmetric and non-symmetric moments

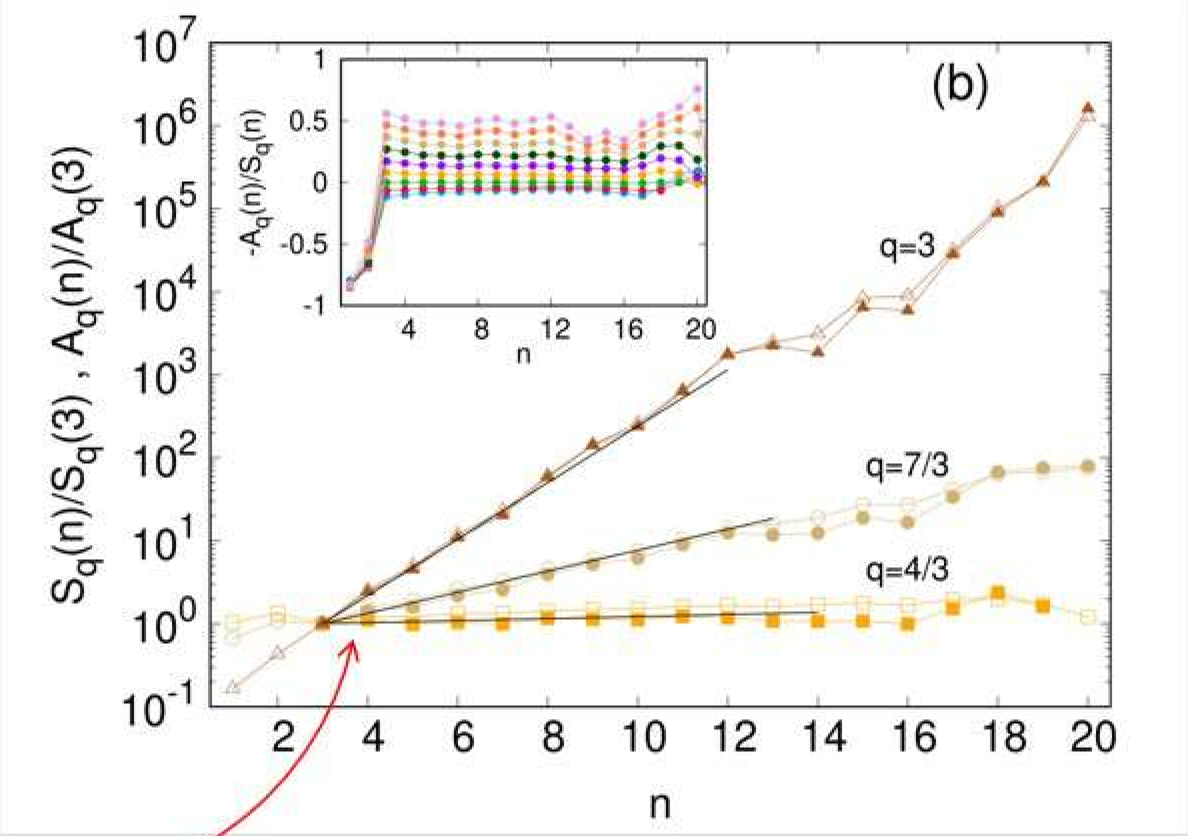
$$S_q(n) = \langle |p_n|^q \rangle / \epsilon^q$$
$$A_q(n) = \langle p_n |p_n|^{q-1} \rangle / \epsilon^q$$

<ins>Scaling law of $\langle |p_n|^q \rangle$:</ins>

(Multifractal prediction)

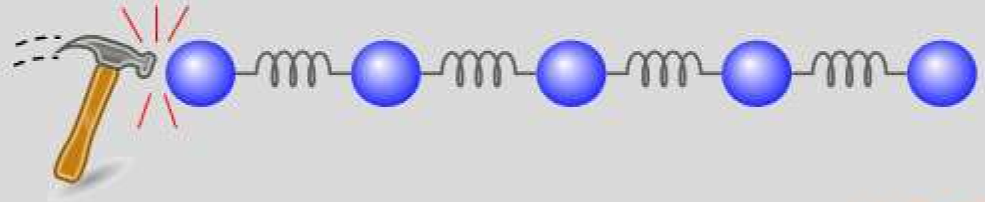
$$\langle p_n^q \rangle \simeq \langle (e_n/t_n)^q \rangle = \langle |u_n|^{2q} \; k_n^q |u_n|^q \rangle \sim k_n^{q-\zeta(3q)}$$

where $\zeta(p)$: $\langle |u_n|^p \rangle \sim k_n^{-\zeta(p)}$



Data and predictions match well

Response to perturbations



Observe average behaviour of a physical quantity B when a physical quantity A is subject to a perturbing force

LINEAR RESPONSE THEORY

(small perturbations)

Response function

$$\langle \delta B(t) \rangle = \int_{t_0}^{t} dt' \, \overbrace{R_{A,B}(t-t')}^{\text{Response function}} \mathcal{F}(t')$$

Kubo's Fluctuation Dissipation Relation (FDR)
(Hamiltonian case)

$$R_{A,B}(t) = -\beta \langle A(t_0) \dot{B}(t) \rangle$$

Generalized FDR
(non-Hamiltonian case, impulsive pert.)

$$R_{x_j,A}(t) = -\left\langle A(\mathbf{x}(t)) \frac{\partial \ln \rho(\mathbf{x})}{\partial x_j} \bigg|_{t=0} \right\rangle$$

when $x_j(0) \to x_j(0) + \Delta x_j$

Chaotic dynamical systems

$$R_{i,j}(\tau) = \frac{1}{M} \sum_{k=1}^{M} \frac{\delta x_j(t_k + \tau | t_k)}{\delta x_i(0)} = \frac{\overline{\delta x_j}(\tau)}{\delta x_i(0)}$$

Response functions in turbulent shell models:
what came before

PHYSICAL REVIEW E, VOLUME 65, 016302

Fluctuation-response relation in turbulent systems

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(Received 29 May 2001; published 14 December 2001)

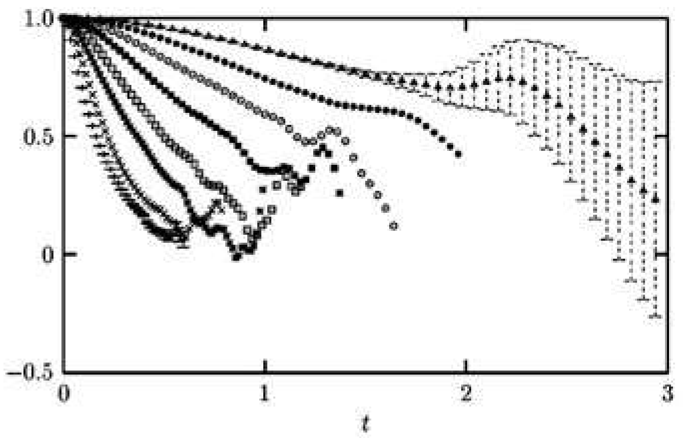

FIG. 4. Modulus of the average response functions, $G_n^n(t) = \langle R_n^n(t) \rangle$, for shells $n=7, \ldots, 14$ (from top to bottom). Error bars are shown only for the smallest and the largest scales. The number of independent kicks used to perform the averages is around 2×10^5. Notice the extremely large error bars measured for the slowest shell variables. The parameters entering in the equations of motion (14) are $b=0.4$, $\nu=5\times 10^{-7}$ for $N=25$ shells.

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **89**, 061002(R) (2014)

Response function of turbulence computed via fluctuation-response relation of a Langevin system with vanishing noise

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(Received 28 November 2013; published 18 June 2014)

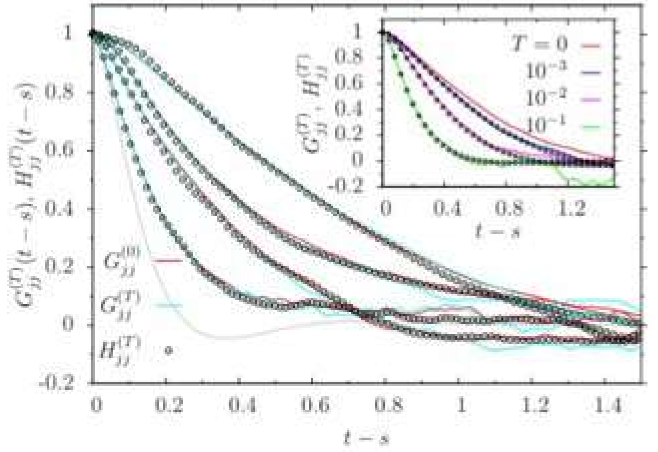


FIG. 2. (Color online) Directly calculated response function of the shell model $G_{jj}^{(0)}$ (zero temperature), $G_{jj}^{(T)}$ with $T=10^{-4}$, and the FRR expression of the response function $H_{jj}^{(T)}$ with $T=10^{-4}$, the right-hand side of Eq. (5), for the shell indices $j=9$, 10, 11, and 12 (from top to bottom). The gray curve is $C_{jj}^{(0)}(t-s)/C_{jj}^{(0)}(0)$ for $j=12$. Inset: Approach of $G_{jj}^{(T)}$ to $G_{jj}^{(0)}$ as $T \to 0$ for $j=9$, plotted with $H_{jj}^{(T)}$.

- <u>Context</u>: Fluctuation-Dissipation Relation in turbulence
- Perturbations on shell velocities
- Small perturbations
- Diagonal response functions only

Energy Response Functions

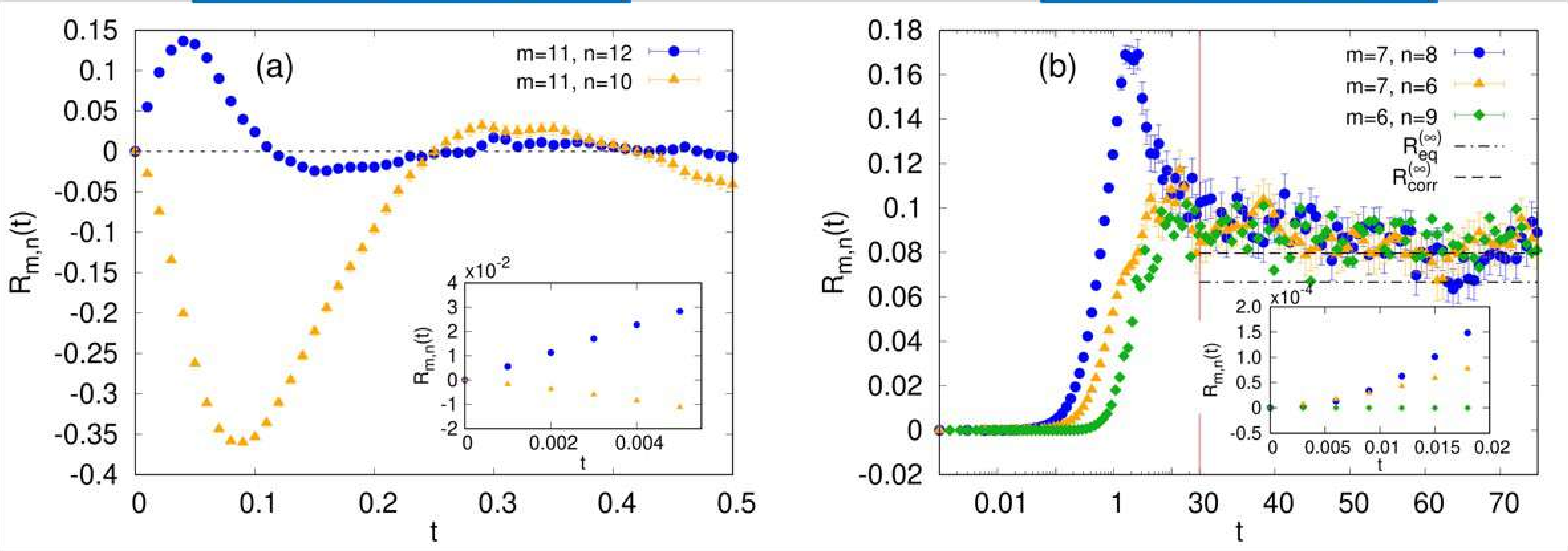
$$R_{\substack{m,n \\ m \neq n}}(t) = \frac{\overline{\delta e_n(t)}}{\delta e_m(0)}$$

Perturbation: $u_m = \sqrt{2e_m}e^{i\theta_m} \quad \rightarrow \quad u'_m = \sqrt{2(e_m + \delta_m)}e^{i\theta_m}$

finite-size perturbation: $\delta_m = f\sigma_{e_m}$ where σ_{e_m}: energy st. dev. $f \simeq 0.2$

TURBULENT $\nu \neq 0,\ f_n \neq 0$

INVISCID $\nu = 0,\ f_n = 0$



Energy Response Functions

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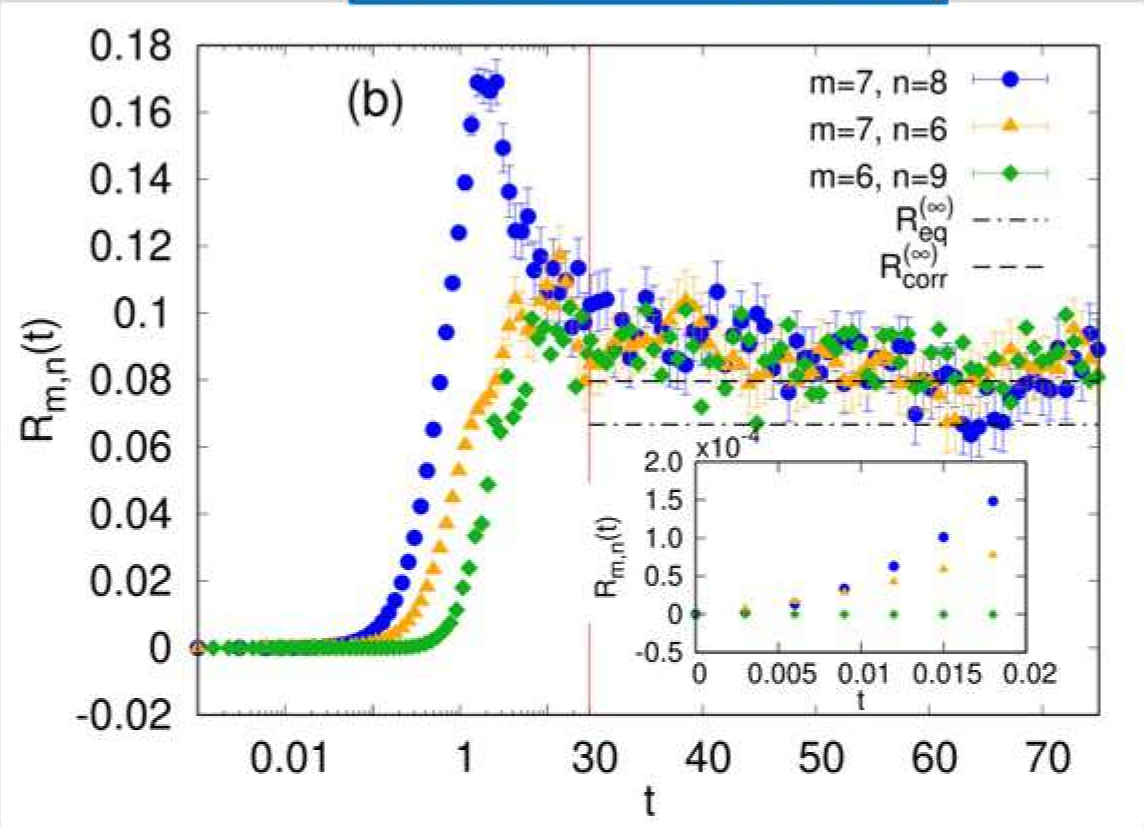

Inviscid shell model ~ truncated Euler equation
(conservative dynamics, statistical equilibrium)

Asymptotic value of $R_{m,n}(t)$:
related to new hypersurface in phase space
where perturbed system evolves

Energy Response Functions

$$R_{m,n}(t) = \frac{\overline{\delta e_n(t)}}{\delta e_m(0)}$$

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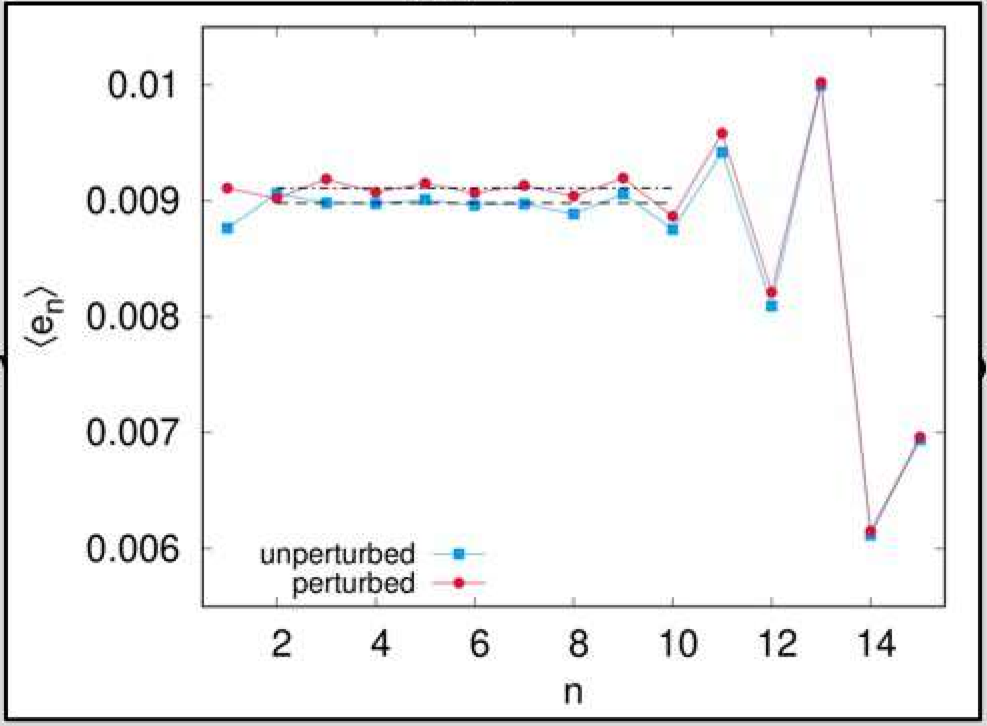
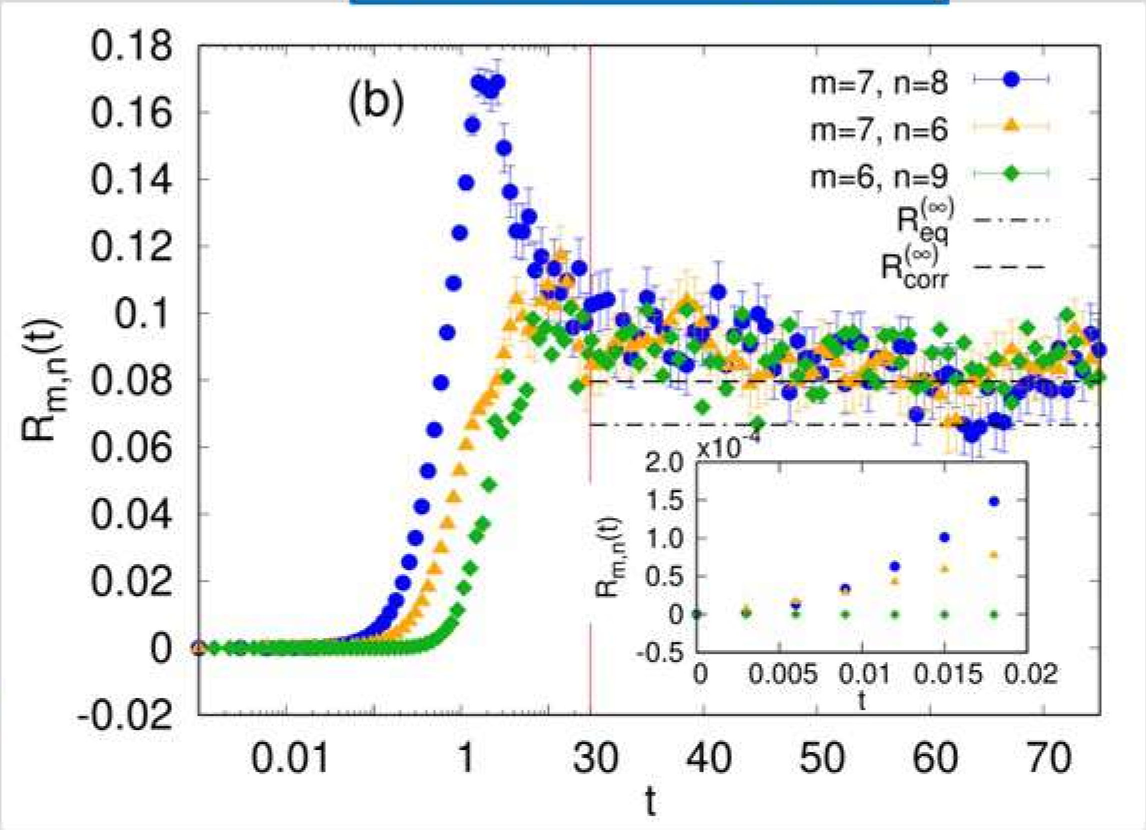


INVISCID $\nu = 0, \; f_n = 0$



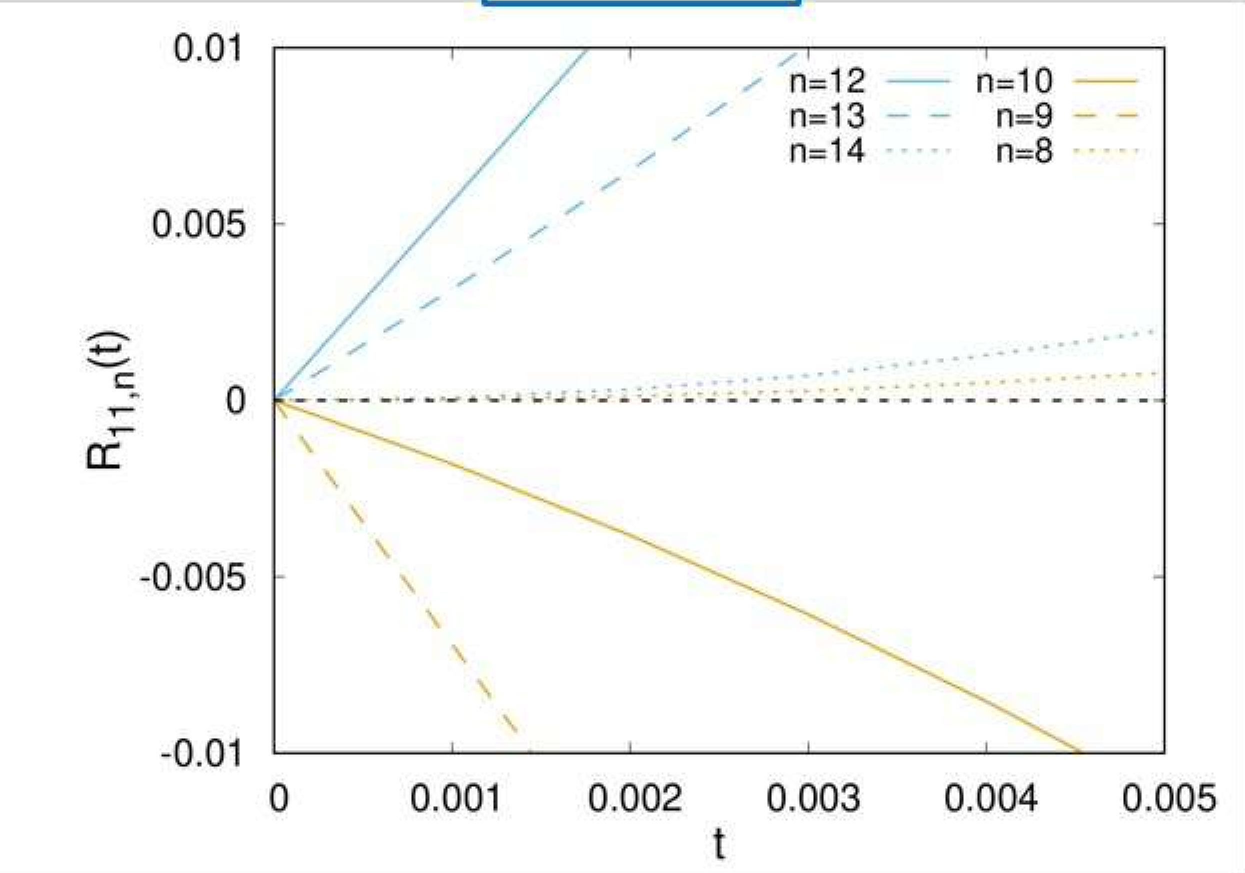
Perfect energy equipartition $\Longrightarrow R_{eq}^{(\infty)} = 1/N$ ✗

Boundary correction $\Longrightarrow R_{corr}^{(\infty)} = (E'_{\text{meas}} - E_{\text{meas}})/N^*$ ✓

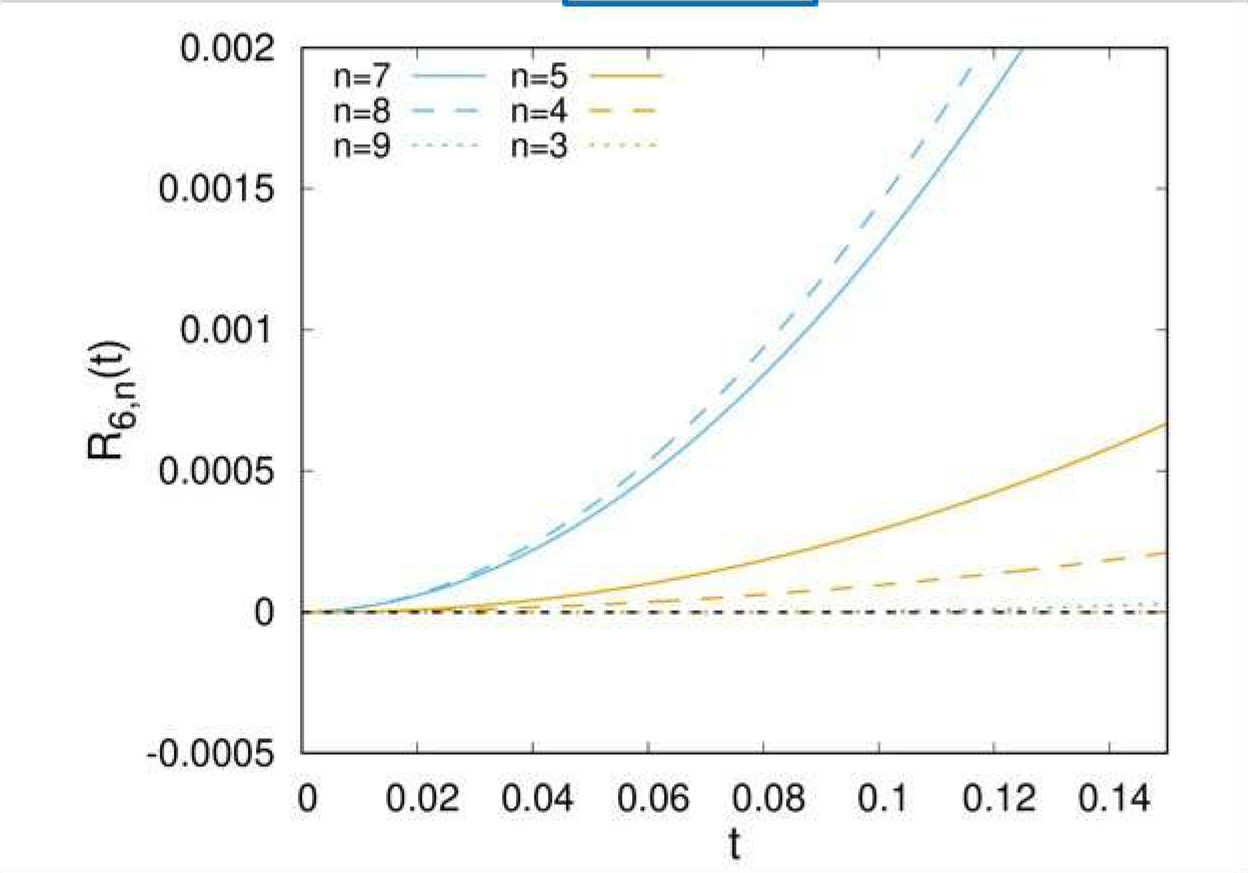
N^* : # equipartited shells

Initial time derivatives

TURBULENT



INVISCID



Initial time derivatives

$$\dot{R}_{m,n}(t)\big|_{t=0} = -\frac{1}{\delta_m}\left[\overline{\delta\Delta_{n+1}} - \tfrac{1}{2}\overline{\delta\Delta_n} - \tfrac{1}{2}\overline{\delta\Delta_{n-1}}\right]\big|_{t=0}$$

in which $\Delta_n = k_n \Im\{u^*_{n-1} u^*_n u_{n+1}\}$

Truncated Euler equation: $P(\boldsymbol{u}) = N\exp\{-\alpha E(\boldsymbol{u}) - \beta H(\boldsymbol{u})\}$

Kraichnan, J Fluid Mech 1973

In shell models the prob. factorizes

$\Downarrow$

Shell velocities are independent random variables

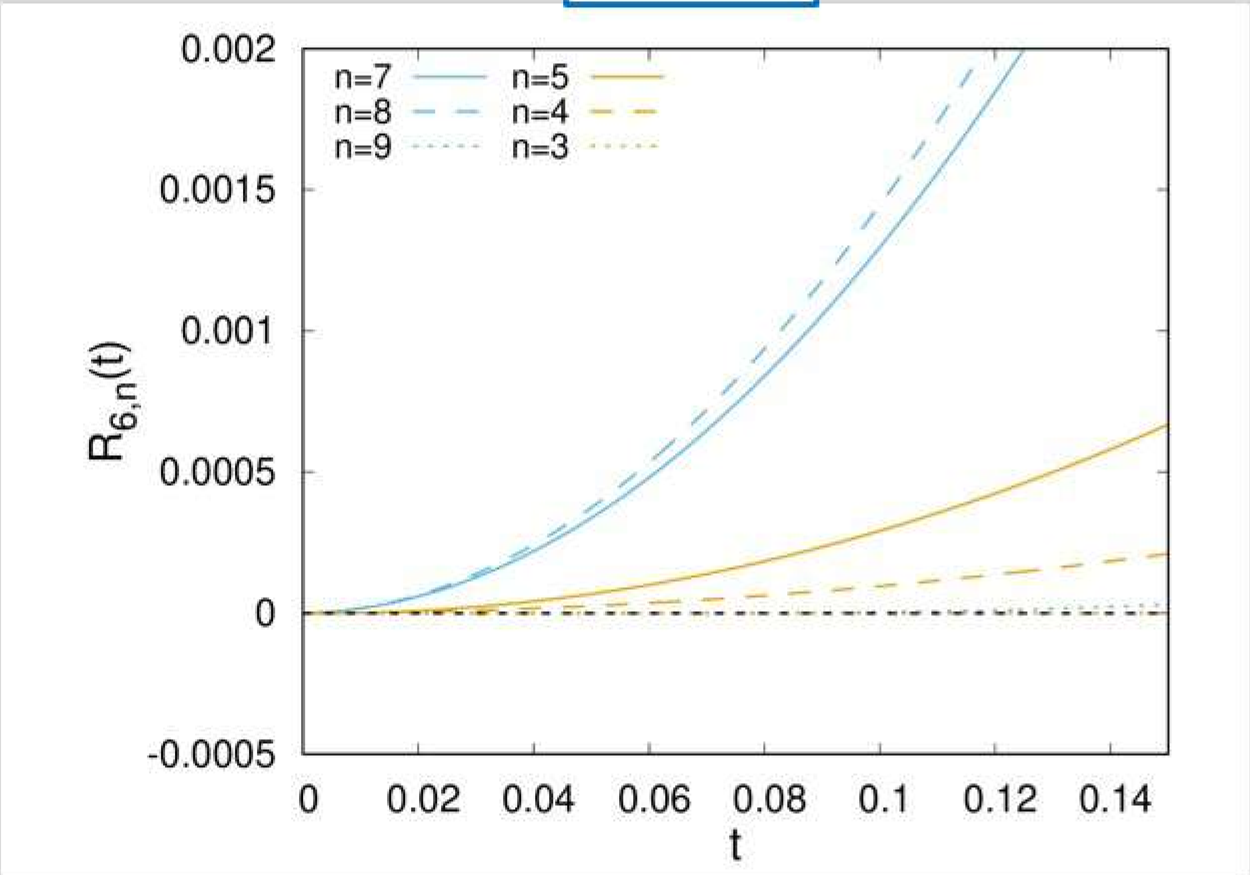
$\Downarrow$

$$\overline{\Delta_n} = 0$$

$\Downarrow$

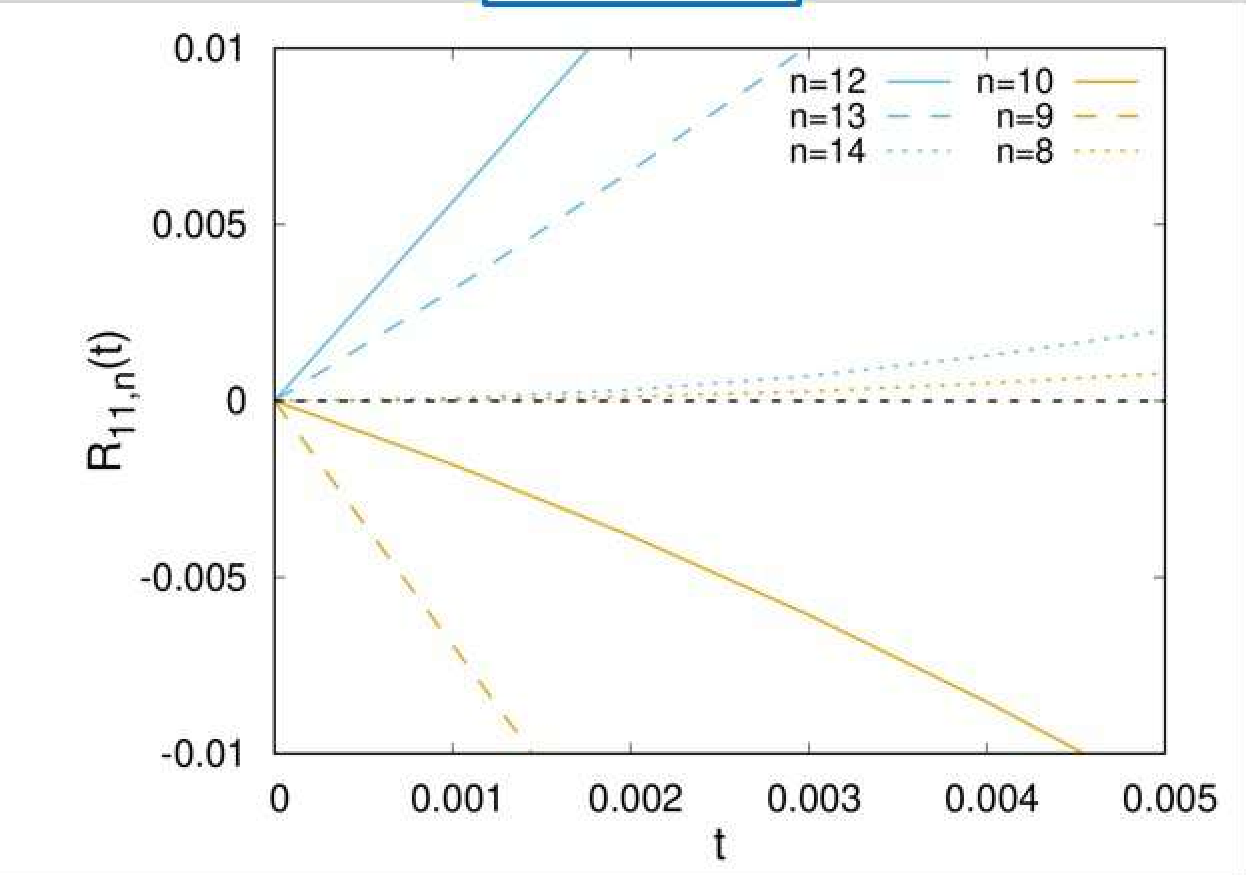
$$\dot{R}_{m,n}(t=0) = 0 \quad \forall m, n$$

INVISCID



Initial time derivatives

TURBULENT



$$\dot{R}_{m,n}(t)\big|_{t=0} = -\frac{1}{\delta_m}\left[\overline{\delta\Delta_{n+1}} - \tfrac{1}{2}\overline{\delta\Delta_n} - \tfrac{1}{2}\overline{\delta\Delta_{n-1}}\right]\Big|_{t=0}$$

in which $\Delta_n = k_n \Im\{u^*_{n-1}u^*_n u_{n+1}\}$

At $t=0$: $\delta u_m = (u'_m - u_m) = \left(\sqrt{1+\delta_m/e_m} - 1\right) u_m = \alpha_m u_m$

$\alpha_m \in \mathbb{R}^+$

$$\Downarrow$$

$$\delta\Delta_n\big|_{t=0} = k_n \Im\left\{\delta_{m,n-1}\alpha_{n-1}u^*_{n-1}u^*_n u_{n+1} + \delta_{m,n}\alpha_n u^*_{n-1}u^*_n u_{n+1} + \right.$$
$$\left. + \delta_{m,n+1}\alpha_{n+1}u^*_{n-1}u^*_n u_{n+1}\right\}$$

In $\dot{R}_{m,n}(t=0)$, if $|m-n| \leq 2$, we have a sum of terms of the kind:

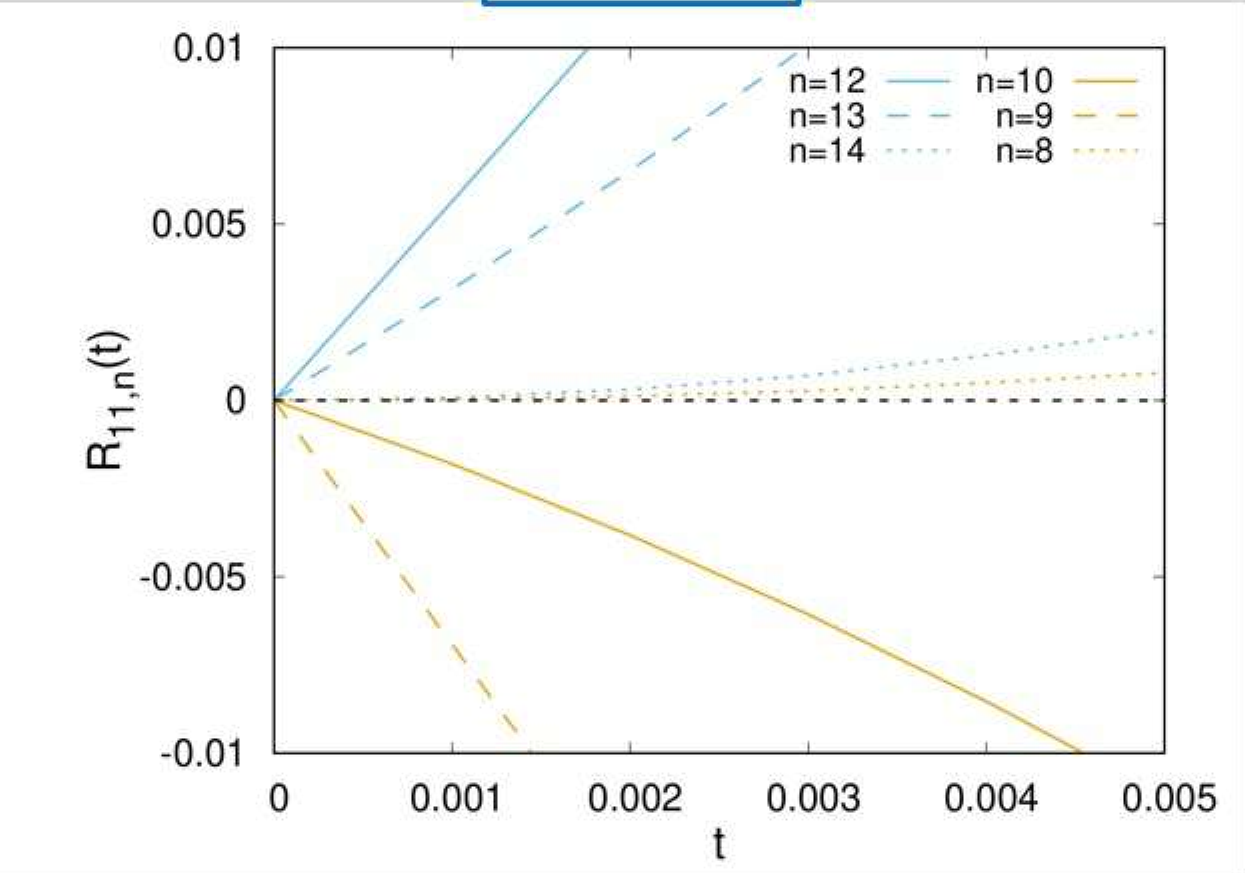
$$\overline{\delta\Delta} = \overline{\alpha\Delta}$$

$\overline{\alpha} > 0$ $\quad$ $\overline{\Delta} > 0$

See eq. (16) in:
L'vov et al., PRE 1998

Initial time derivatives

TURBULENT



$$\dot{R}_{m,n}(t)\big|_{t=0} = -\frac{1}{\delta_m}\left[\overline{\delta\Delta_{n+1}} - \tfrac{1}{2}\overline{\delta\Delta_n} - \tfrac{1}{2}\overline{\delta\Delta_{n-1}}\right]\Big|_{t=0}$$

in which $\Delta_n = k_n \Im\{u^*_{n-1}u^*_n u_{n+1}\}$

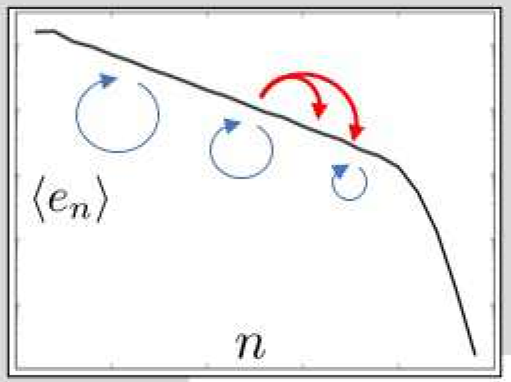
n	$\dot{R}_{m,n}(t=0)$
$m-2$	$-\overline{\alpha_m \Delta_{m-1}}\big\|_{t=0} < 0$
$m-1$	$\overline{\alpha_m \left[\frac{1}{2}\Delta_{m-1} - \Delta_m\right]}\big\|_{t=0} < 0$
$m+1$	$\overline{\alpha_m \left[\frac{1}{2}\Delta_m + \frac{1}{2}\Delta_{m+1}\right]}\big\|_{t=0} > 0$
$m+2$	$\overline{\alpha_m \frac{1}{2}\Delta_{m+1}}\big\|_{t=0} > 0$

See eq. (16) in:
L'vov et al., PRE 1998

Instead $\dot{R}_{m,n}(t=0) = 0$ if $|m-n| > 2$

(too far to feel initial perturbation)

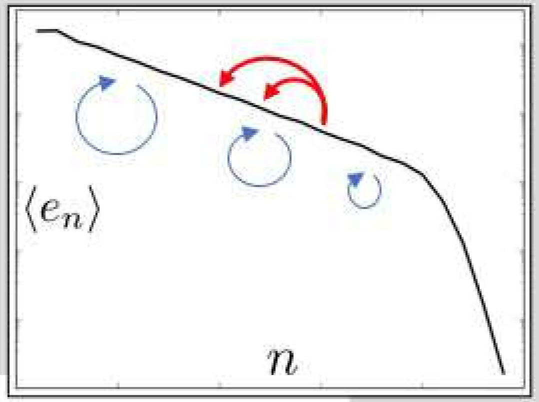
Turbulent response:
the complete picture

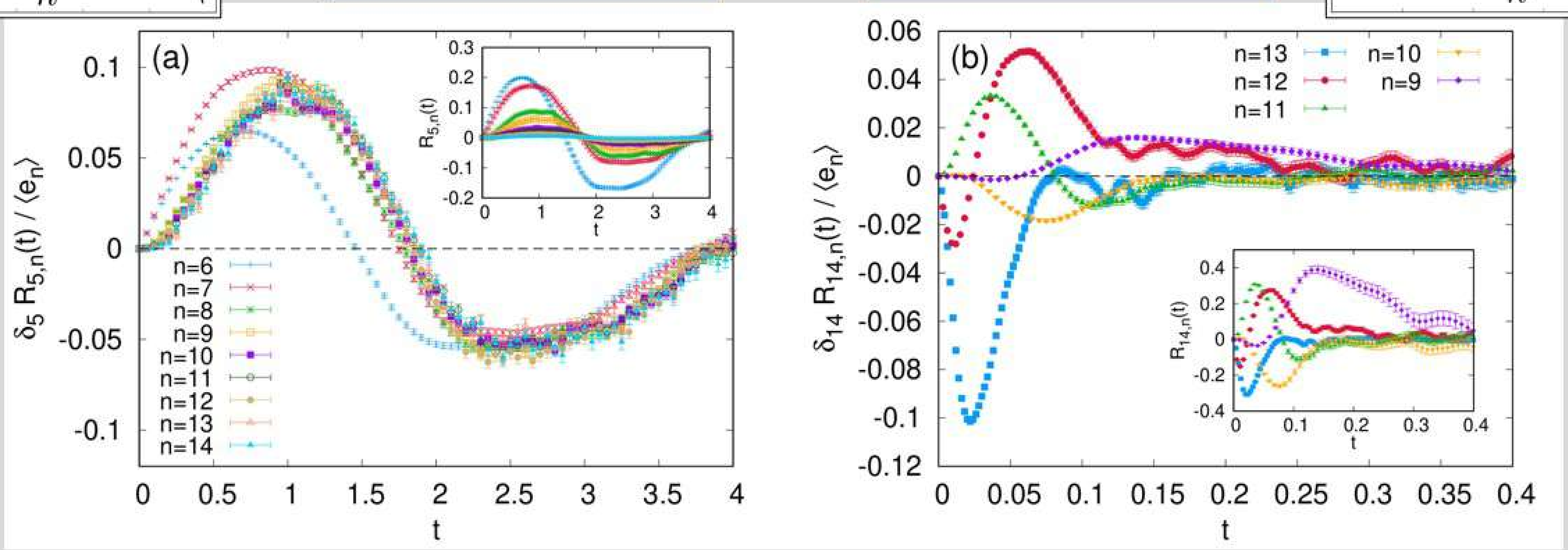


$$\frac{\delta_m}{\langle e_n \rangle} R_{m,n} = \frac{\overline{\delta e_n}(t)}{\langle e_n \rangle}$$

"Forward" rescaled RFs

"Backward" rescaled RFs





coherent, non-damped behaviour

incoherent, damped behaviour

Intermediate-scale forcing

Are the modes larger than forcing scale in statistical equilibrium?

Simulations and experiments: yes

Recent results: no, deviation from eq.

PRL 115, 204501 (2015) PHYSICAL REVIEW LETTERS week ending 13 NOVEMBER 2015

Statistical Equilibria of Large Scales in Dissipative Hydrodynamic Turbulence

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(Received 8 July 2015; published 11 November 2015)

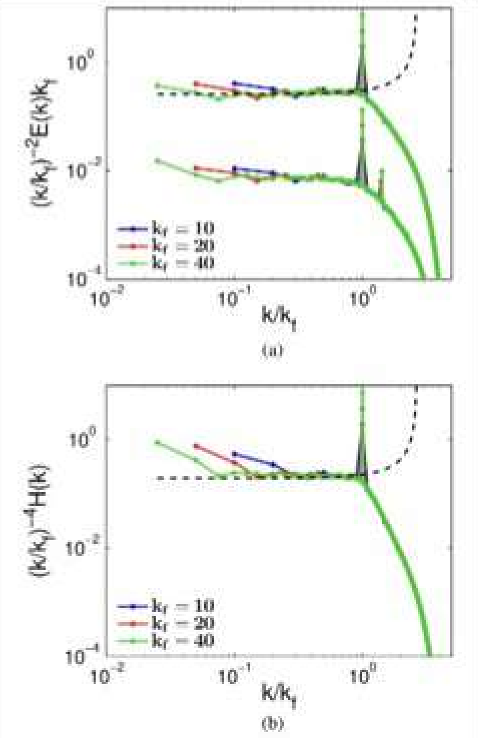

FIG. 2 (color online). (a) Compensated $k^{-2}E(k)$ energy spectra for helical (top) and nonhelical (bottom) flows. (b) Compensated $k^{-4}H(k)$ helicity spectra. The dotted lines represent Kraihnan's absolute equilibria [Eqs. (1)].

PHYSICAL REVIEW LETTERS 129, 054501 (2022)

Statistical Equilibrium of Large Scales in Three-Dimensional Hydrodynamic Turbulence

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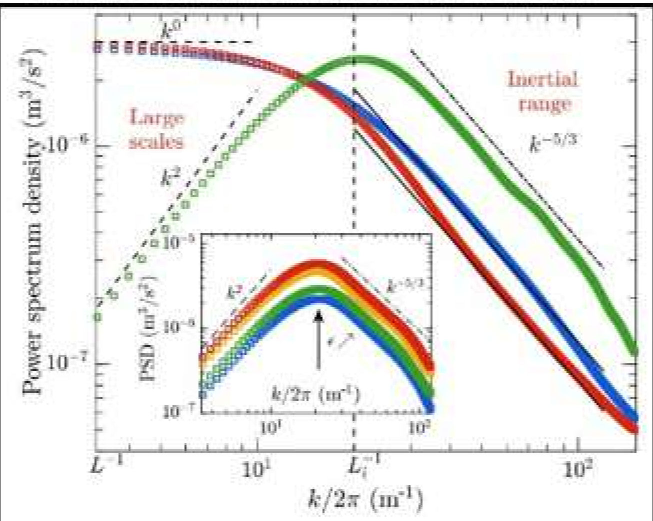

FIG. 1. 3D power spectrum density $E(k)$ (green) derived from the 1D spectra of the longitudinal velocity $E_{uu}(k_x)$ (red), and transverse velocity $E_{vv}(k_x)$ (blue), [Eq. (4)]. Dashed line: k^2 power law illustrating the large-scale statistical equilibrium regime. Dot-dashed line: $k^{-5/3}$ power law illustrating the inertial range of the turbulent cascade. The vertical dashed line corresponds to the inverse of the integral scale $k_i/2\pi = 1/L_i$ and separates the large-scale domain ($k < k_i$) from the inertial range ($k > k_i$). The PIV measurements are performed at $F = 20$ Hz, $B = 290$ G and $N = 55$. Inset: power spectrum densities (PSD) at different $\epsilon \in [1.1, 3.2] \times 10^{-4}$ m²/s³.

Intermediate-scale forcing

Are the modes larger than forcing scale in statistical equilibrium?

Simulations and experiments: yes — Recent results: no, deviation from eq.

Departure from the statistical equilibrium of large scales in three-dimensional hydrodynamic turbulence

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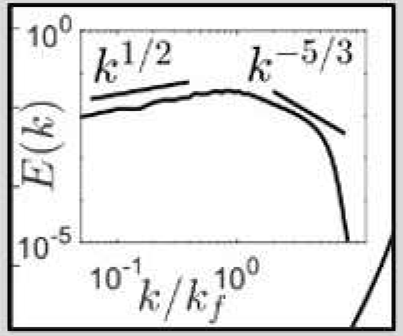
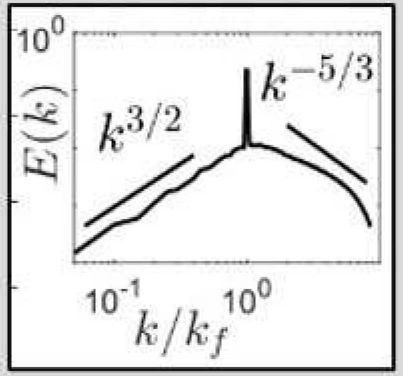


Exp. correlated forcing



Spherical shell forcing

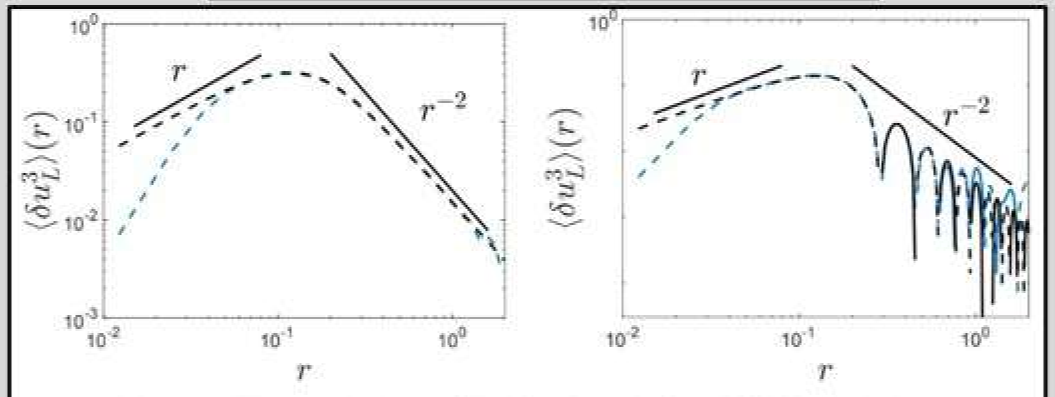


Figure 2: Theoretical solution (black) and results from DNS (blue) for the longitudinal third-order structure function $\langle \delta u_L^3 \rangle$. The left and right panels represent the exponential (type I) forcing and spherical shell (type II) forcing, respectively. Solid lines represent the positive values, and dashed lines represent the absolute value of the negative values.

KHM equation

3D HIT, large scales:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^4\langle \delta u_L^3 \rangle)\right) = -6\underbrace{\langle \mathbf{F}\cdot\mathbf{u}' + \mathbf{F}'\cdot\mathbf{u}\rangle}_{=0 \quad \text{if } r > r_F}$$

$$\Downarrow$$

$$\langle \delta u_L^3 \rangle \sim r^{-2}$$

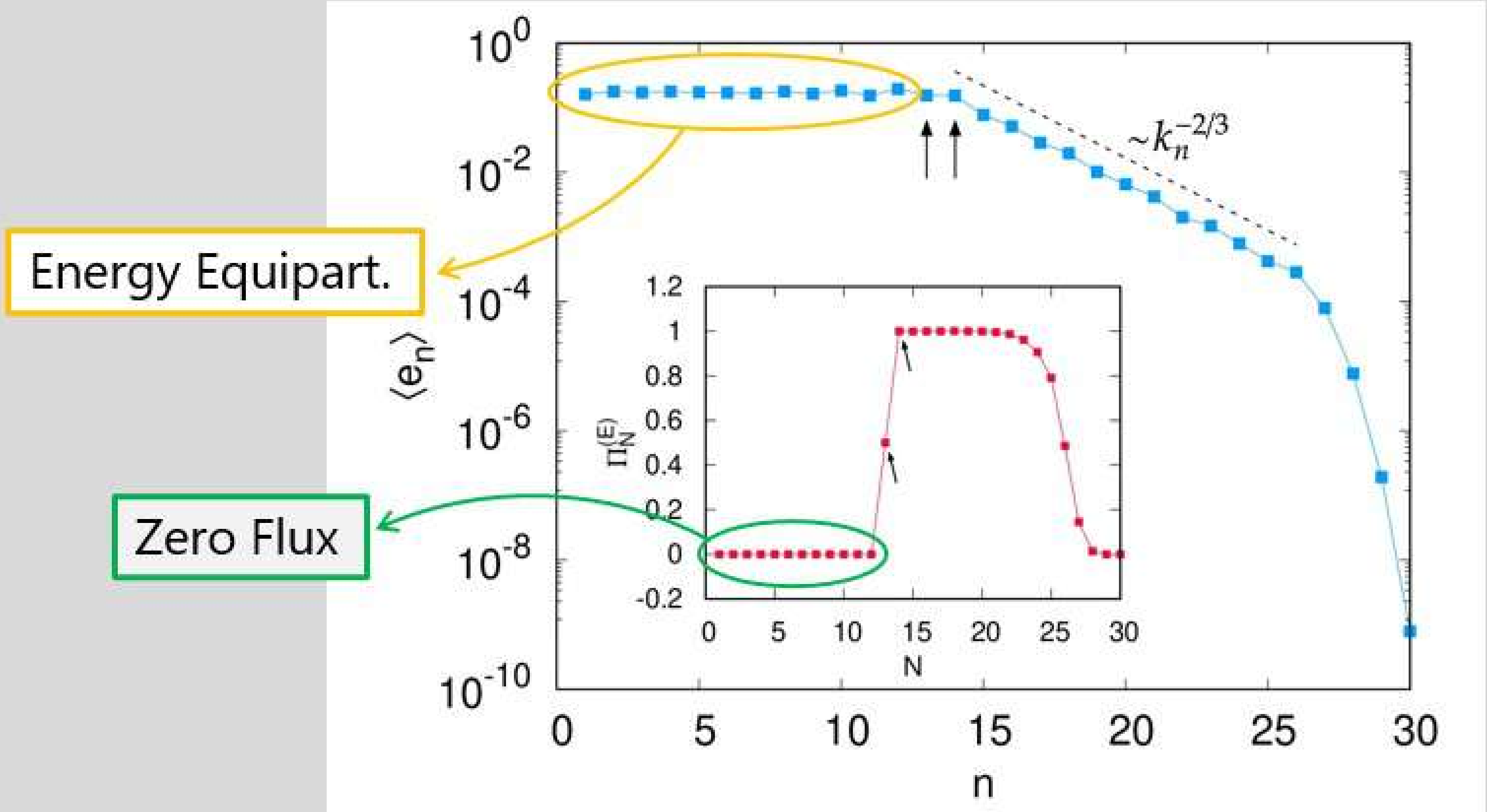
(if equilibrium: ~ 0)

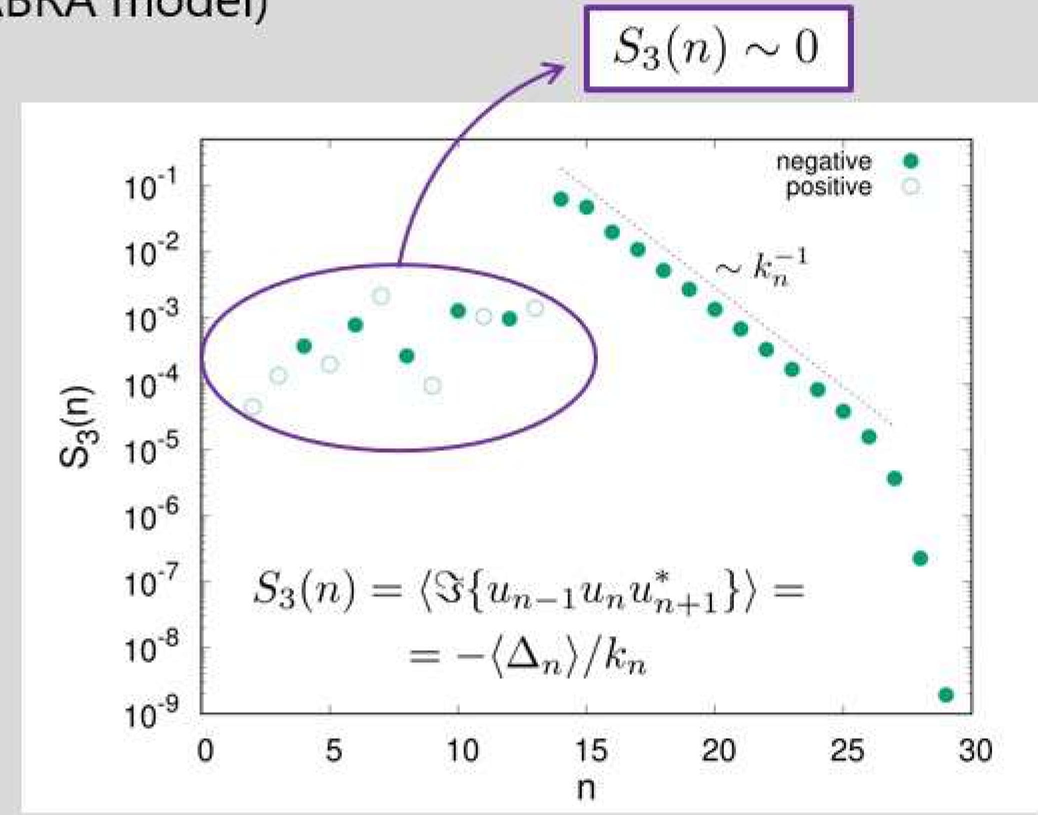
Intermediate-scale forcing

Are the modes larger than forcing scale in statistical equilibrium?

Simulations and experiments: yes — Recent results: no, deviation from eq.

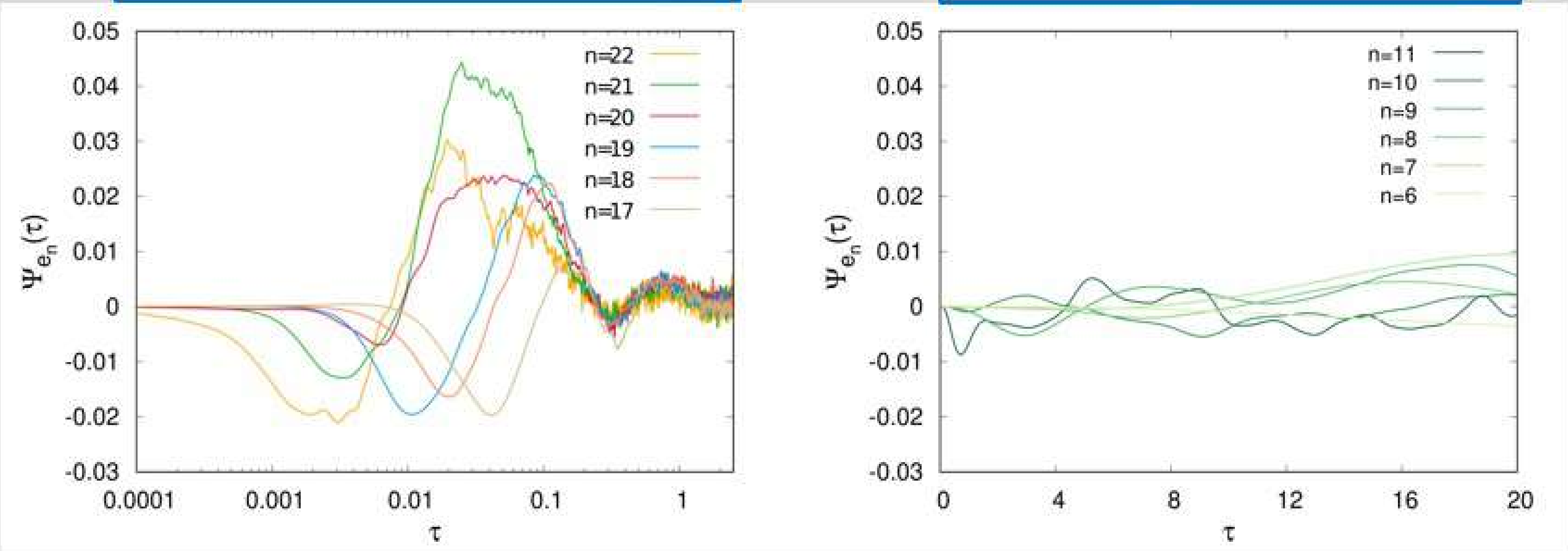
Our findings (SABRA model)

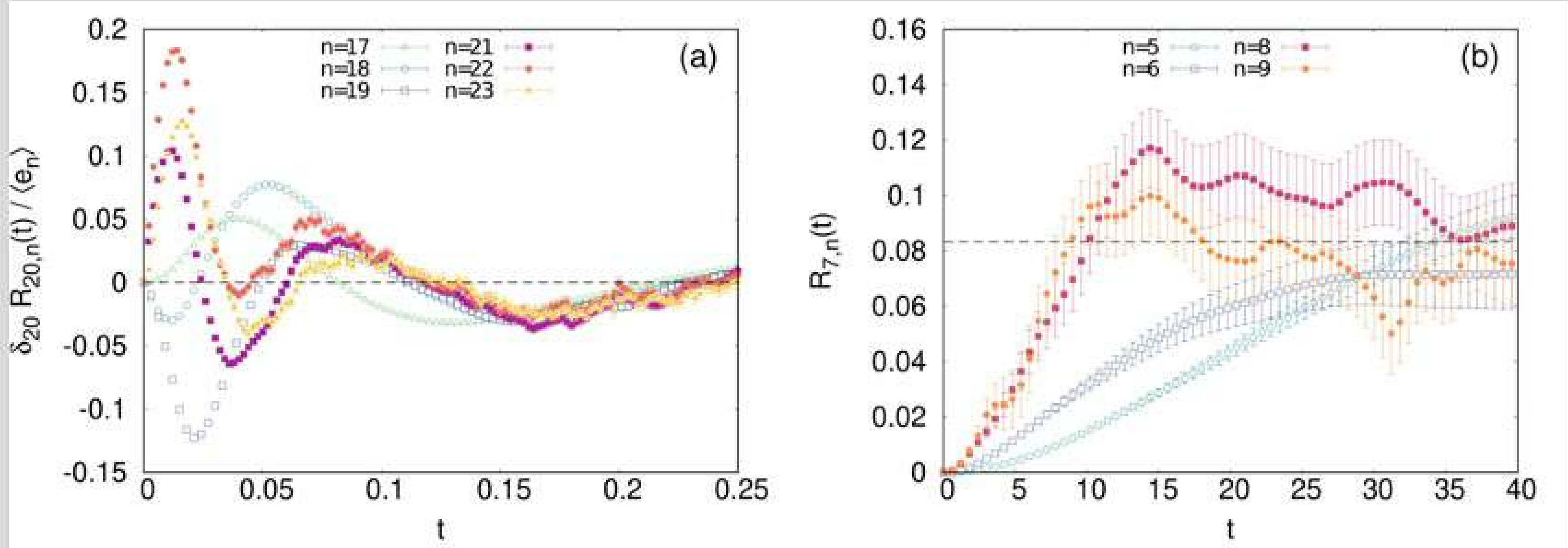




Intermediate-scale forcing

Scales smaller than forcing | **Scales larger than forcing**





Time correlations:
at large scales, small random fluctuations around zero

Response functions:
at large scales, positive asymptotic value like inviscid case

Conclusions

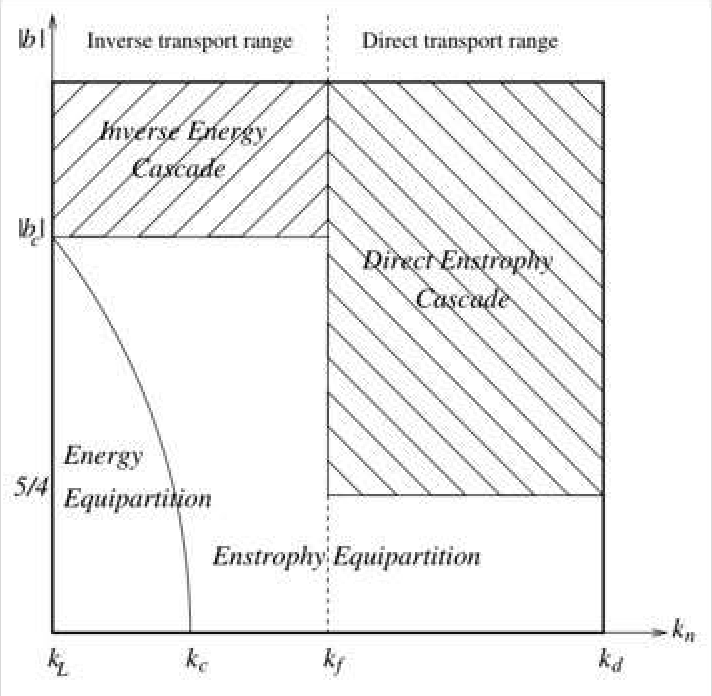
1. Time-correlation and response functions are able to discriminate between inviscid and turbulent systems
2. Time-irreversibility in 3D turbulence is linked to time-asymmetric energy transfers among different length scales
3. Response to energy perturbation reveals the presence and the 'direction' of the energy cascade
4. Scales larger than the forced ones seem to be in agreement with a statistical equilibrium picture (at least in shell models)

Reference

N. Cocciaglia, M. Cencini, A. Vulpiani, '**Non-equilibrium statistical mechanics of the turbulent energy cascade: irreversibility and response functions**', arXiv:2310.07470 (2023)

Future developments

1. SABRA shell model in 2D
   - different phenomenology of the cascade
     (inverse energy cascade, direct enstrophy cascade)
   - cascade and equilibrium give same scaling:
     distinction is more subtle



Gilbert et al., PRL 2002

2. DNS (especially for intermediate-scale forcing case)
   - no restriction to local interaction as in shell model
   - time correlations: problem of sweeping