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Time-analyticity of Lagrangian particle trajectories in ideal fluid flow governed by the Euler equations

Prof. Uriel Frisch

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(P.I. Prof. Luca Biferale)

Università degli Studi di Roma Tor Vergata
C.F. n. 80213750583 – Partita IVA n. 02133971008 - Via della Ricerca Scientifica, 1 – 00133 ROMA

Time-analyticity of Lagrangian particle trajectories in ideal fluid flow governed by the Euler equations

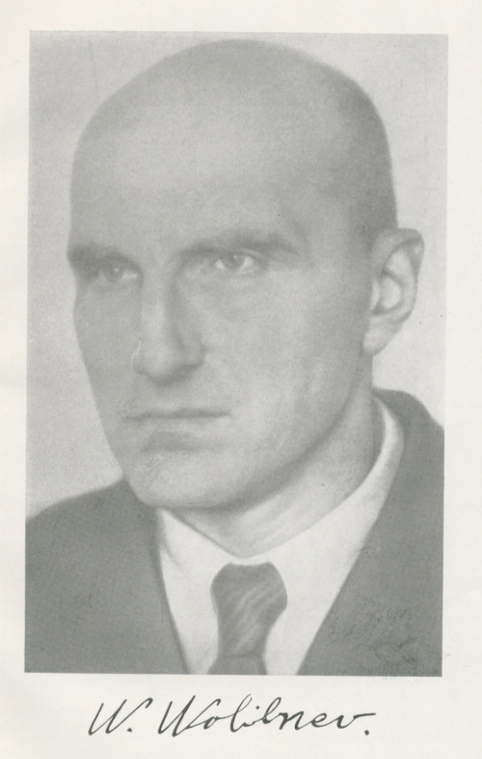
Uriel Frisch
Laboratoire Joseph-Louis Lagrange
Observatoire de la Côte d'Azur, Nice

Part II: Using Cauchy's invariants to prove time-analyticity of Lagrangian trajectories

arXiv:1212.4333 and 1312.6320 [math.AP] (*with V. Zheligovsky and O. Podvigina)*

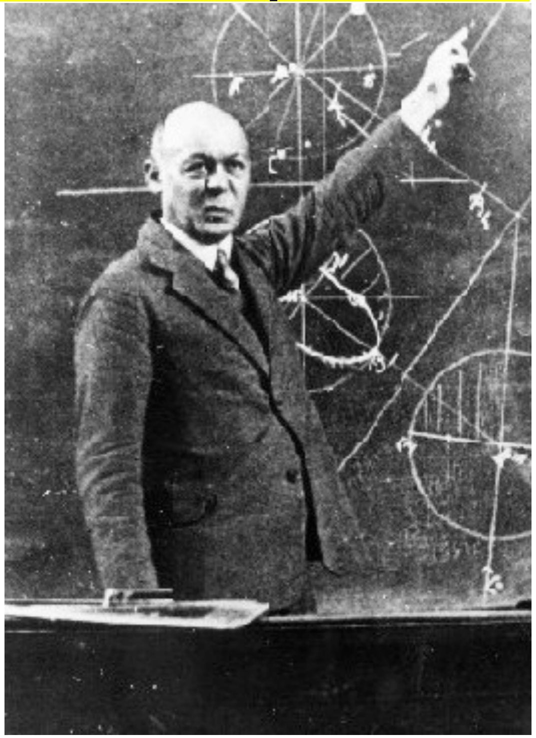
Perspectives: Lagrangian-Eulerian numerical simulations of blowup for 3D Euler

The birth of functional analysis for the Euler equations



Witold Wolibner
1902-1961
(Warsaw Polytech. Inst.)

The first proof of persistence for some time of initial regularity for the solutions of the 3D Euler equations was given by Lichtenstein (1927) using tools introduced by Hölder. This effort was continued by Witold Wolibner, Ernst Hölder, who proved all-time regularity for 2D in 1933 and many others.



Leon Lichtenstein 1878-1933
(U. Leipzig: 1922-1933)

Lichtenstein assumed that the initial vorticity satisfies an "H-condition", i.e. is Hölder continuous. For space-periodic solutions to the 3D Euler equations, Lichtenstein's key estimate reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{\omega}(t)|_\alpha < C|\boldsymbol{\omega}(t)|_\alpha^2 \qquad 0<\alpha<1, \quad C>0 \qquad |\boldsymbol{\omega}|_\alpha := \max_{\boldsymbol{x}\in\mathbb{T}^3}|\boldsymbol{\omega}(\boldsymbol{x})| + \sup_{\boldsymbol{x}_1,\,\boldsymbol{x}_2\in\mathbb{T}^3,\,\boldsymbol{x}_1\neq\boldsymbol{x}_2}\frac{|\boldsymbol{\omega}(\boldsymbol{x}_1)-\boldsymbol{\omega}(\boldsymbol{x}_2)|}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^\alpha}$$

Such solutions are said here to have "limited" regularity.

A very smooth ride in a rough sea

- The limited-regularity solutions of Lichtenstein and his followers have a $C^{1,\alpha}$ regularity in space (their first derivatives are Hölder continuous). In Eulerian coordinates, their temporal regularity is also not better than $C^{1,\alpha}$. The Lagrangian structure, i.e. the trajectories of fluid particles, is however much smoother.
- Chemin (1982) showed that during the interval of regularity they remain *indefinitely differentiable* in time.
- Serfati (1995) and Shnirelman (2012), using the theory of analytic functions on complex Banach spaces, showed that the trajectories of fluid particles are actually analytic in time. Their proofs are difficult and do not give estimates of the radius of convergence of the time-Taylor series for the Lagrangan map.
- Frisch and Zheligovsky (2014), using Cauchy's formulation proved the following *Theorem.* Consider a space-periodic three-dimensional flow of incompressible fluid governed by the Euler equation. Suppose the initial velocity $\boldsymbol{v}_0(\boldsymbol{a})$ is in $C^{1,\alpha}(\mathbb{T}^3)$. Then, at sufficiently small times, the position of fluid particles, $\boldsymbol{x}(\boldsymbol{a},t)$, is given by a temporal Taylor series whose coefficients can be recursively calculated. The radius of convergence is bounded from below by a strictly positive quantity, which is inversely proportional to $|\nabla \boldsymbol{v}_0|_{0,\alpha}$.

$$\sum_{k=1}^{3} \nabla^{\mathrm{L}} \dot{x}_k \times \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0$$

$$\det(\nabla^{\mathrm{L}} \boldsymbol{x}) = 1$$

Introduce the displacement: $\boldsymbol{\xi} := \mathbf{x} - \mathbf{a}$

$$\nabla^{\mathrm{L}} \times \dot{\boldsymbol{\xi}} + \nabla^{\mathrm{L}} \dot{\xi}_k \times \nabla^{\mathrm{L}} \xi_k = \boldsymbol{\omega}_0$$

$$\det\,(\mathbf{I} + \nabla^{\mathrm{L}} \boldsymbol{\xi}) = 1 \quad \text{or} \quad \nabla^{\mathrm{L}} \cdot \boldsymbol{\xi} + \frac{1}{2}\Big[(\nabla^{\mathrm{L}} \cdot \boldsymbol{\xi})^2 - \mathrm{tr}\big(\nabla^{\mathrm{L}} \boldsymbol{\xi}\big)^2\Big] + \det(\nabla^{\mathrm{L}} \boldsymbol{\xi}) = 0$$

Expand (formally) in powers of t: $\boldsymbol{\xi} = \sum_{n=1}^{\infty} t^n \boldsymbol{\xi}^{(n)}$, and determine coefficients of various powers

$$n \nabla^{\mathrm{L}} \times \boldsymbol{\xi}^{(n)} + \sum_{r+s=n} r \nabla^{\mathrm{L}} \xi_k^{(r)} \times \nabla^{\mathrm{L}} \xi_k^{(s)} = \boldsymbol{\omega}_0 \delta_{n1}, \quad n = 1, 2, \ldots$$

$$\nabla^{\mathrm{L}} \cdot \boldsymbol{\xi}^{(n)} + \frac{1}{2} \sum_{r+s=n} \Big[\nabla^{\mathrm{L}} \cdot \xi^{(r)}\, \nabla^{\mathrm{L}} \cdot \xi^{(s)} - \mathrm{tr}\big(\nabla^{\mathrm{L}} \xi^{(r)} \nabla \xi^{(s)}\big)\Big] + \sum_{r+s+\sigma=n} \nabla^{\mathrm{L}} \xi_1^{(r)} \cdot \big(\nabla^{\mathrm{L}} \xi_2^{(s)} \times \nabla^{\mathrm{L}} \xi_3^{(\sigma)}\big) = 0$$

Solve for $\boldsymbol{\xi}^{(n)}$ (Helmholtz-Hodge decomposition). Define $\zeta_n := |\nabla \boldsymbol{\xi}^{(n)}|_{0,\alpha}$.

Use the boundedness of $\Delta^{-1}\nabla\nabla$ in Hölder spaces (Korn 1907, Lichtenstein 1925, Stein 1970,...):

$$\zeta_n \le C_1 |\mathbf{v}_0|_{1,\alpha} \delta_{n1} + C_2 \sum_{r+s=n} \zeta_r \zeta_s + C_3 \sum_{r+s+\sigma=n} \zeta_r \zeta_s \zeta_\sigma, \quad n = 1, 2, \ldots \quad C_1 > 0, \quad C_2 > 0, \quad C_3 > 0$$

Elementary proof of analyticity (2)

Introduce the generating function $F(t) := \sum_{n=1}^{\infty} t^n \zeta_n$

$$F(t) \le C_1 t |\mathbf{v}_0|_{1,\alpha} + C_2 F^2(t) + C_3 F^3(t), \quad t > 0$$

$F(t)$ bounded for $0 \le t |\nabla \mathbf{v}_0|_{0,\alpha} < \tau_\star$

where $\tau_\star > 0$ is such that the discriminant of the cubic equation

$C_1 \tau_\star - F + C_2 F^2 + C_3 F^3 = 0$ vanishes.

This follows from the observation that the polynomial

$P(F) := C_3 F^3 + C_2 F^2 - F$

has a finite local maximum, a finite local minimum and two positive roots, colliding on increasing t.

Analyticity follows from the boundedness of the generating function.

QED

Analytic continuation; radius of convergence of the Lagrangian time-Taylor series as blowup indicator

- Typically, the Lagrangian map is analytic but not entire in time: it has a finite radius of convergence, R (even in 2D).

- For any $0 < t < R$, one can construct a new time-Taylor series for a Lagrangian map, whose radius of convergence is $R(t)$.

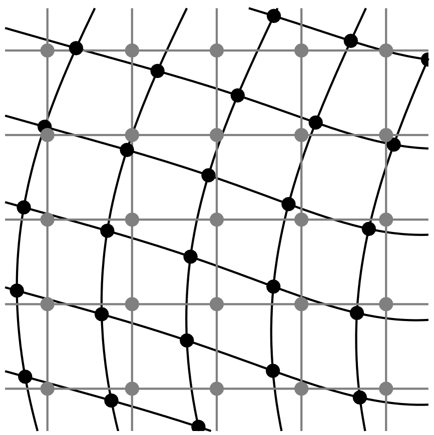
- One can iterate this procedure and construct a sequence of increasing times $0 < t_1 < t < \ldots < t_m < \ldots$. This can be continued as long as $R(t_m)$ does not vanish.

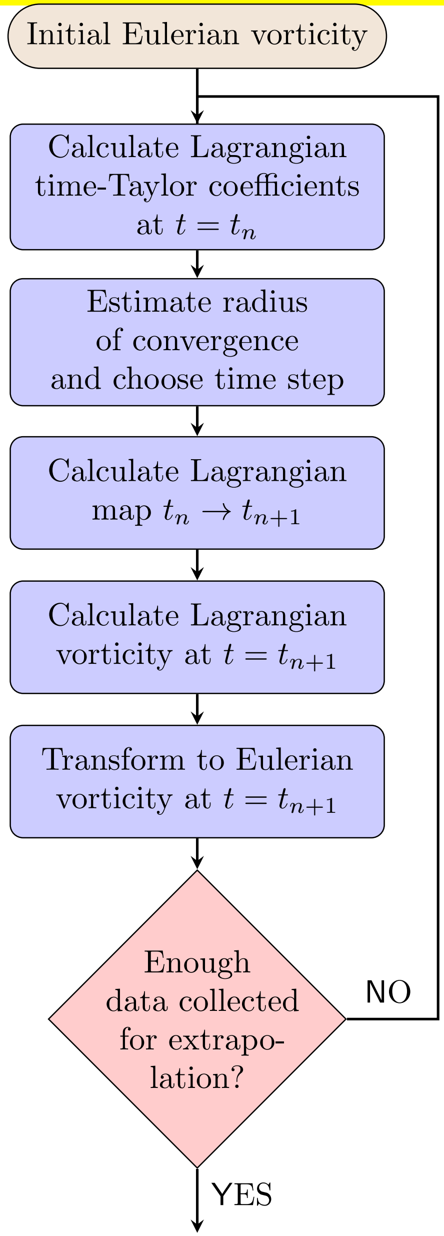
- The vanishing of the radius of convergence indicates a blowup.

The Cauchy-Lagrangian algorithm built on the Cauchy invariants

- In 1928 Courant, Friedrichs and Lewy showed that numerical solutions of hyperbolic PDE's by simple finite difference methods are subject to the constraint $\Delta t < \Delta x / V_{\max}$.

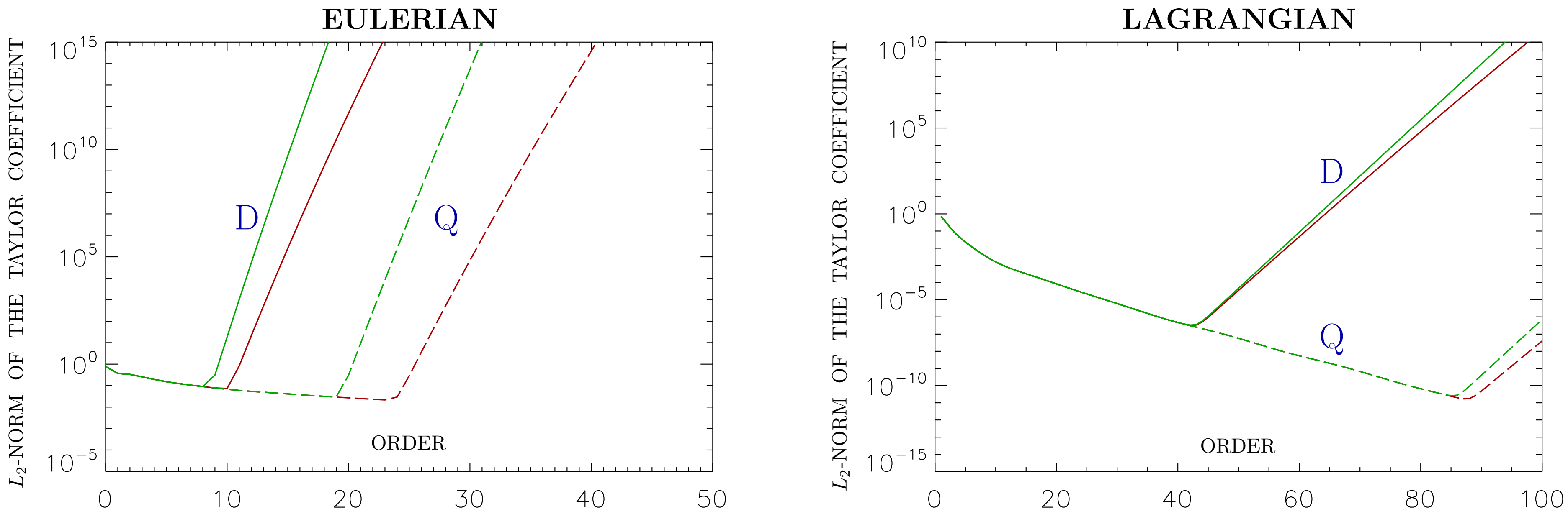
- In hydrodynamics, this affects Eulerian but not Lagrangian algorithms. The use of Lagrangian time-Taylor expansions allows us to study blowup numerically.





Eulerian - but not Lagrangian - computations suffer from loss of smoothness, implying dramatic loss of precision

- In Eulerian coordinates, if $\mathbf{v} \in C^{k,\alpha}$, then $\partial_t \mathbf{v} \in C^{k-1,\alpha}$
- In Lagrangian coordinates, if $\mathbf{v} \in C^{k,\alpha}$, then $D_t \mathbf{v} \in C^{k,\alpha}$



D: double, Q: quadruple precision. Resolution: 512^2, 1024^2

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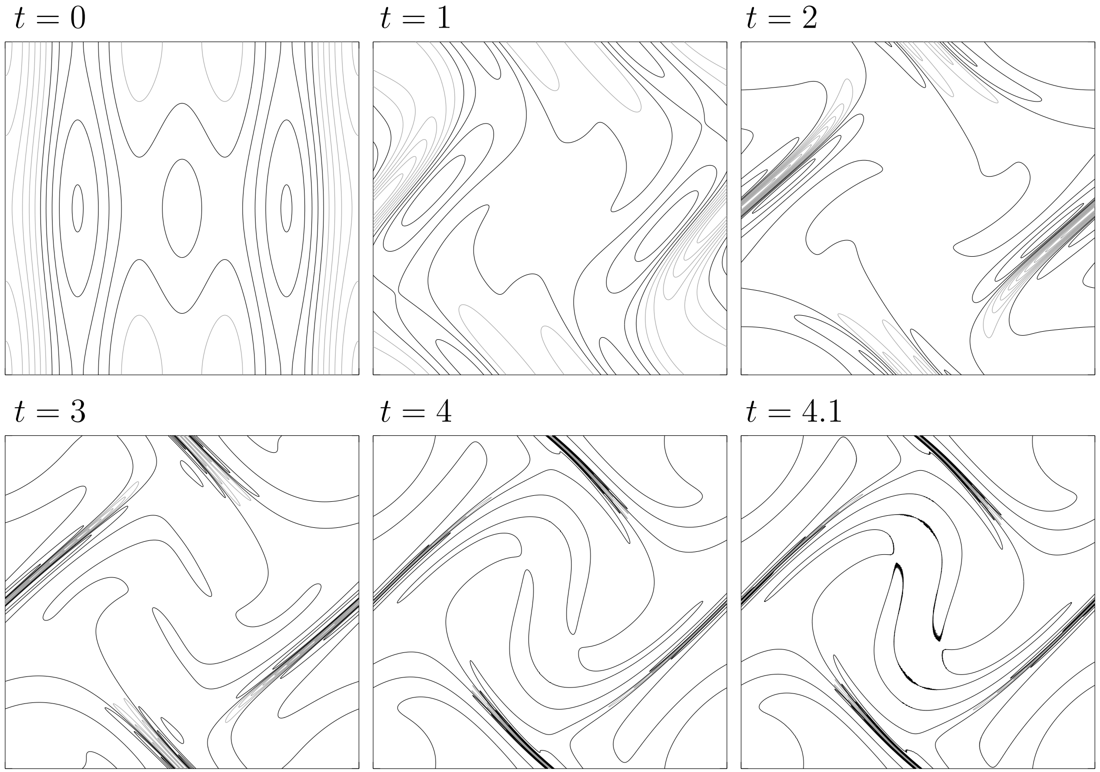
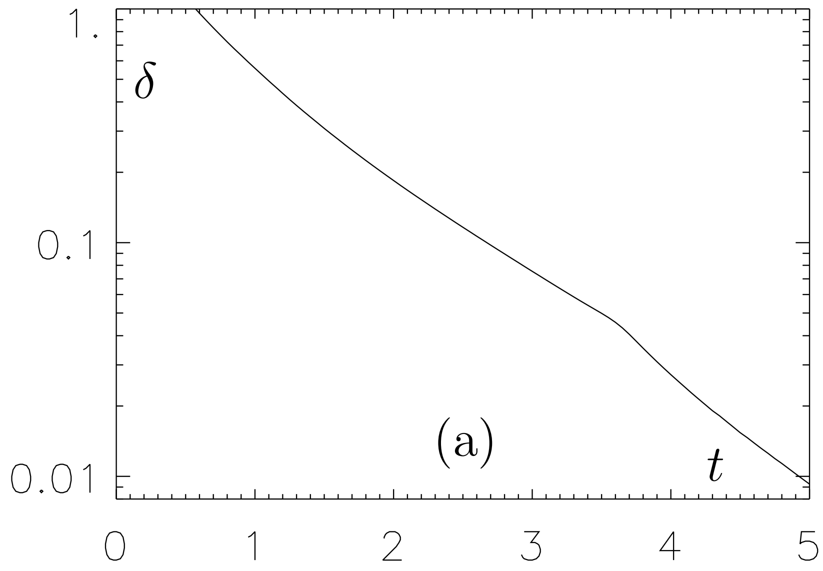
Switching from Eulerian to Lagrangian computations can result in speed up of several orders of magnitude

$$\boldsymbol{\omega}^{(\mathrm{init})} = \cos x + \cos y + 0.6\cos 2x + 0.2\cos 3x$$



Isolines of the Laplacian of vorticity

Distance $\delta(t)$ to the real domain of the nearest complex-space singularity



Resolution: 8192^2 harmonics

Speed up CL20/RK4 : 120