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Slide of the Seminar

On the multifractal structure of fully developed turbulence

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On the multifractal structure of fully developed turbulence

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Navier -Stokes equation

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta v_i$$

$$\frac{\partial v_i}{\partial x_i} = 0$$

hence $\Delta p = -\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \rightarrow$ "nonlocal" nonlinearity

Eulerian structure functions

- Eulerian transverse structure functions:

$$S_n^{\perp}(l) = \left\langle \left| (\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})) \times \frac{\mathbf{l}}{l} \right|^n \right\rangle \propto l^{\zeta_n^{\perp}}$$

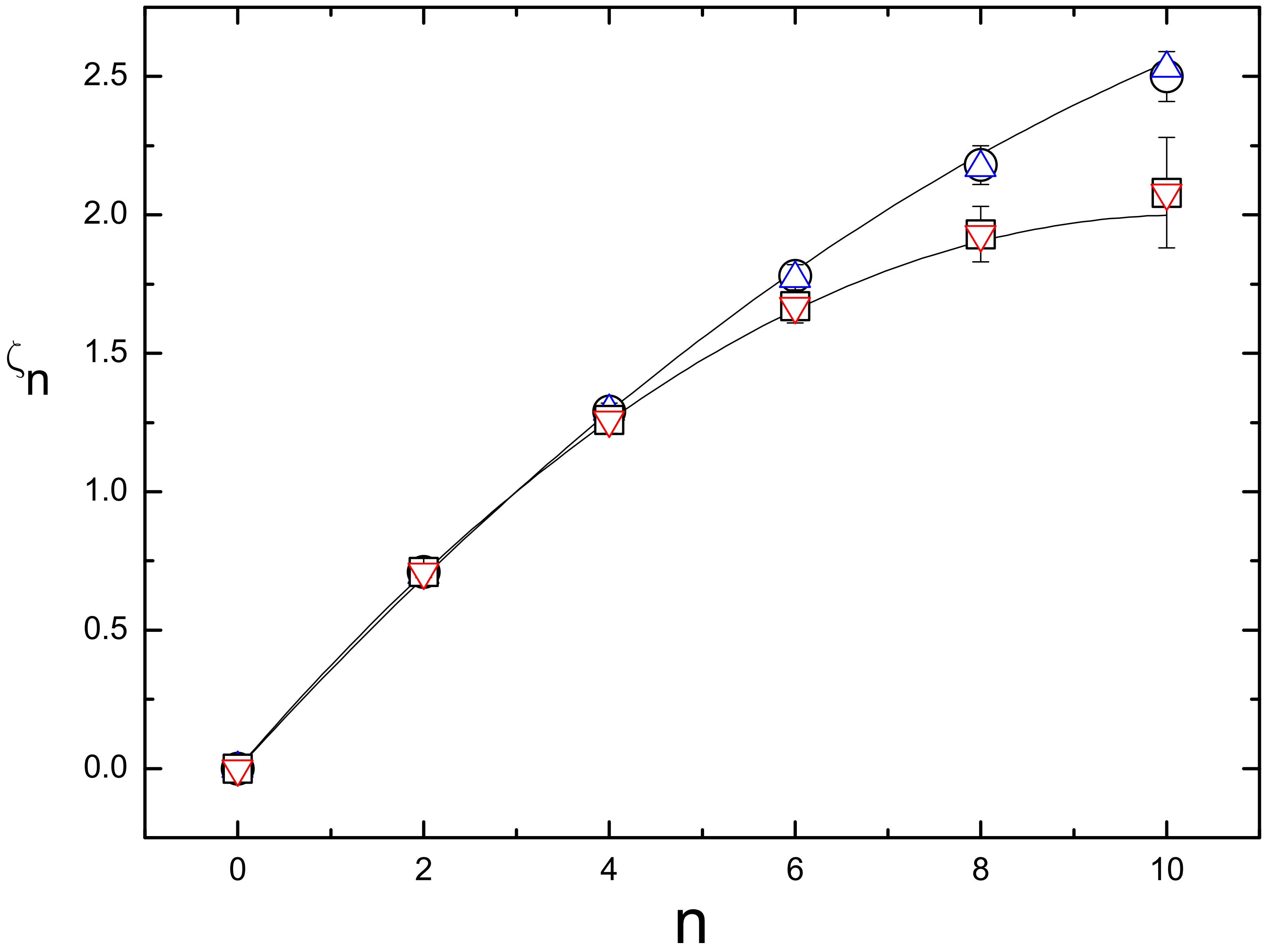
- Eulerian longitudinal structure functions:

$$S_n^{||}(l) = \left\langle \left| (\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})) \cdot \frac{\mathbf{l}}{l} \right|^n \right\rangle \propto l^{\zeta_n^{||}}$$

- Modern experiment and numerical calculations $S_n : \ n \sim 8 - 10$
- there is **no** theory based on Navier Stokes equation
- exact result $\zeta_{2,3}^{\perp} = \zeta_{2,3}^{||}$
- theoretical expectations: $\zeta_n^{||} = \zeta_n^{\perp}$

Numerical simulations (Benzi et al. 2010, Gotoh et al. 2002)

$\zeta_n^{\parallel} \neq \zeta_n^{\perp}$ **poor accuracy ?**



Kolmogorov (K41) theory

- stationary, locally isotropic and homogeneous turbulence in incompressible fluid
- inertial range – dimensional theory – cascade

$$\eta \ll l \ll L\text{– Eulerian case}$$

- structure functions

$$\zeta_n^E = n/3\text{– Eulerian case}$$

experiment – anomalous scaling

Modern numerical simulations (M.Farge)

Total vorticity

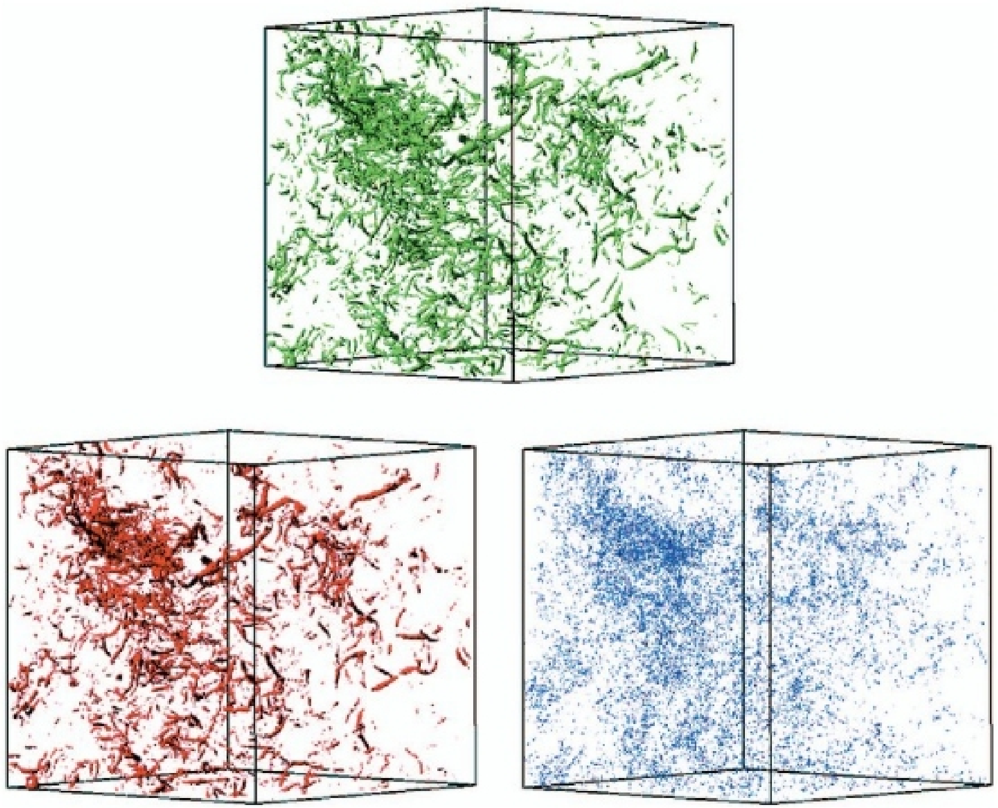
Coherent

Vorticity

2.6% N coefficients

80% enstrophy

99% energy



Incoherent

Vorticity

94.7% N coefficients

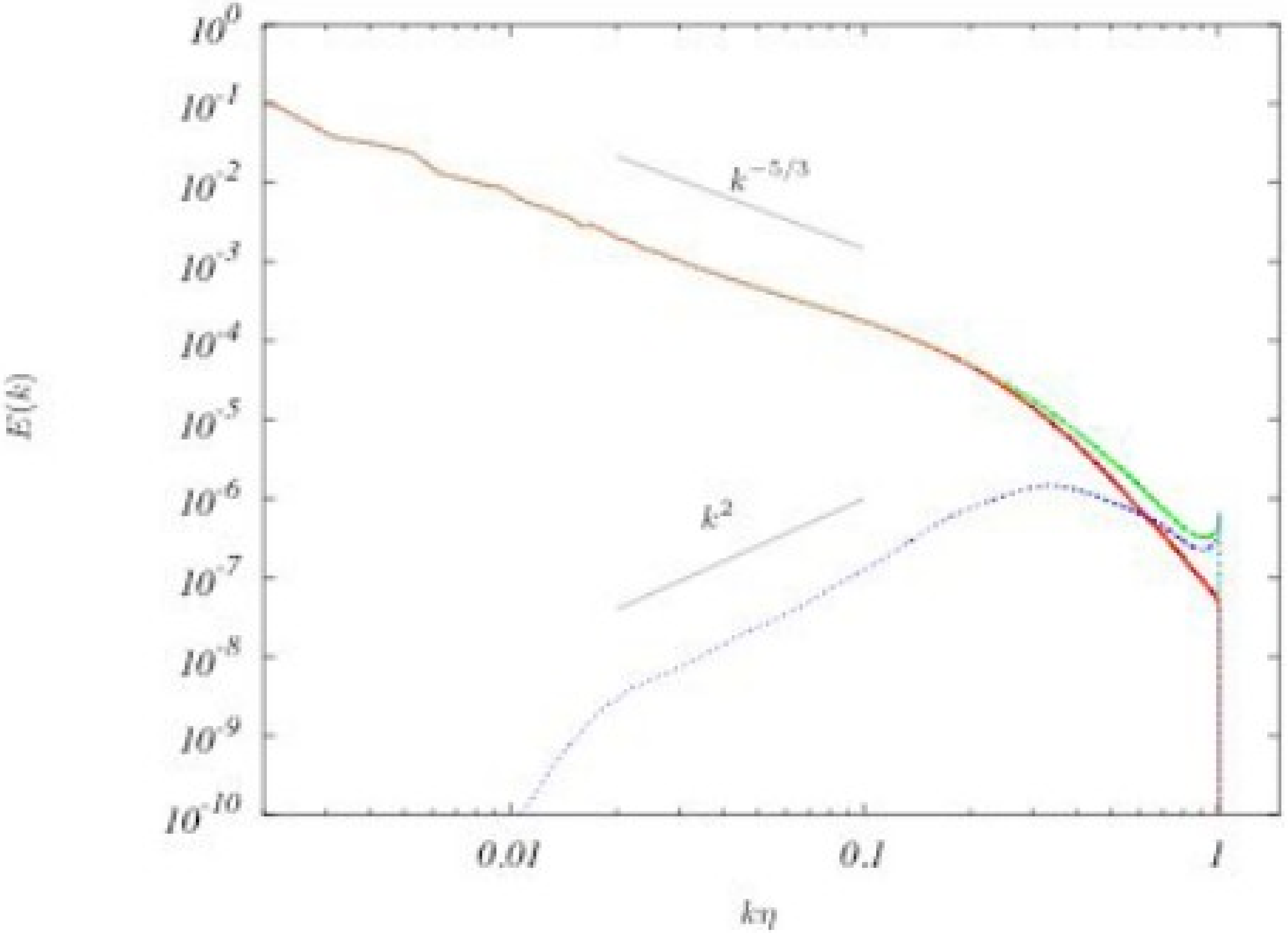
20% enstrophy

1% energy

vortex filaments –99% energy! 80% dissipation

life-time 100 τ_c

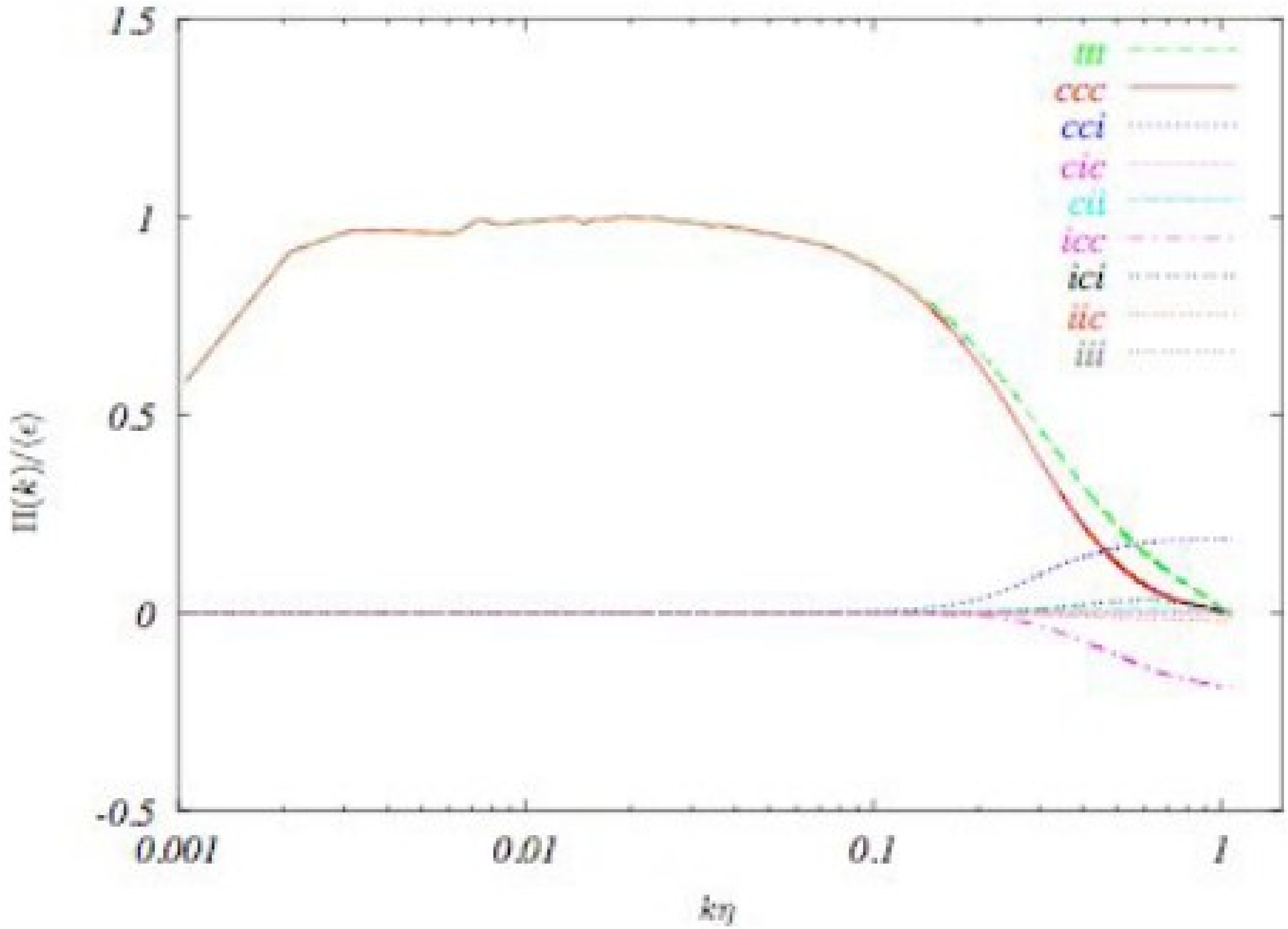
Okamoto et al., 2007 Phys. Fluids, 19(11)



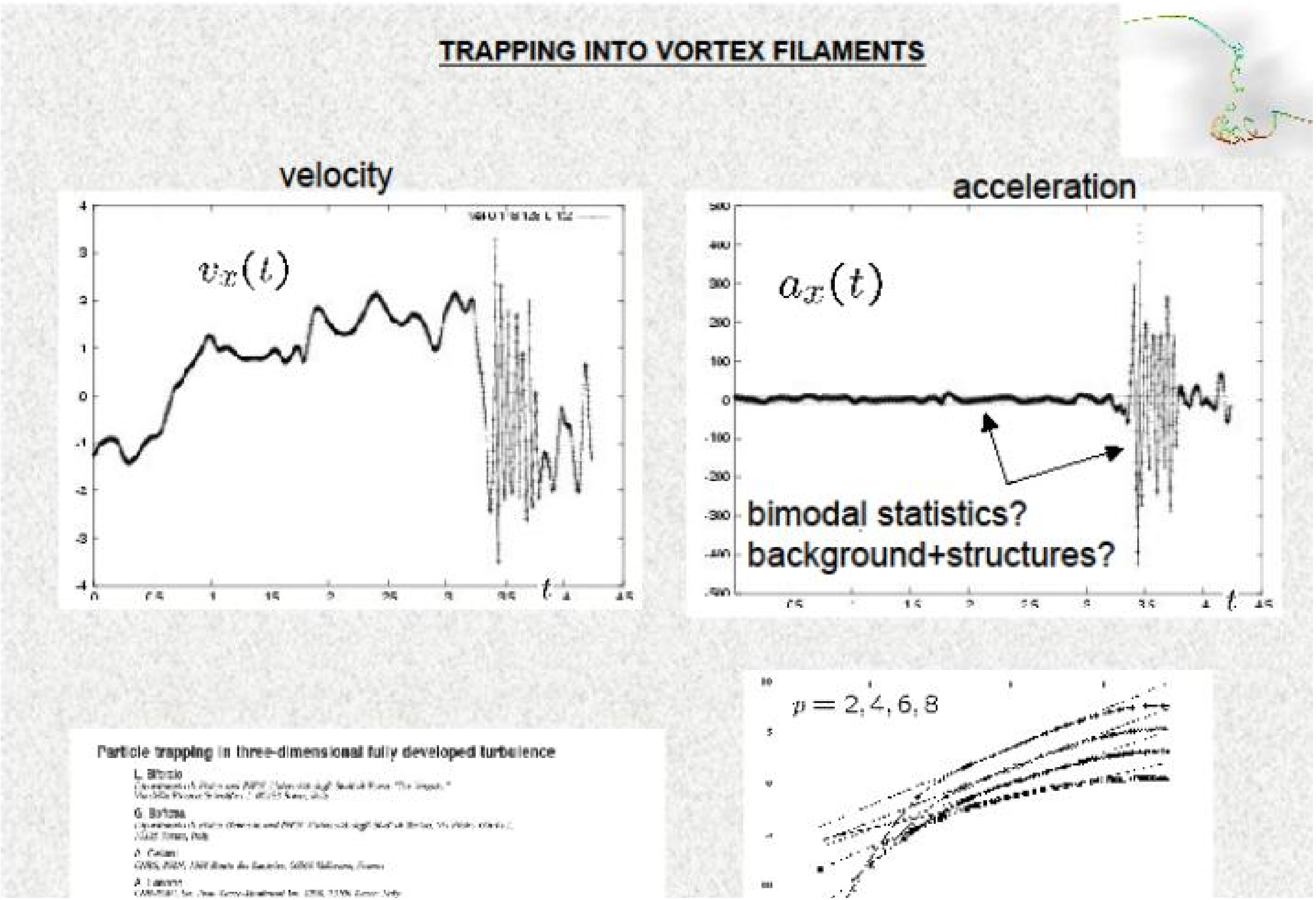
Energy flux

incoherent input = 0

Okamoto et al., 2007 Phys. Fluids, 19(11)

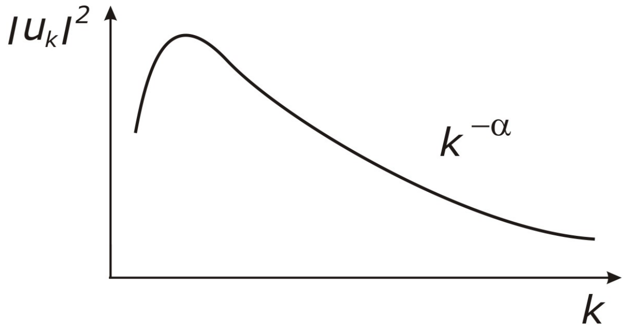


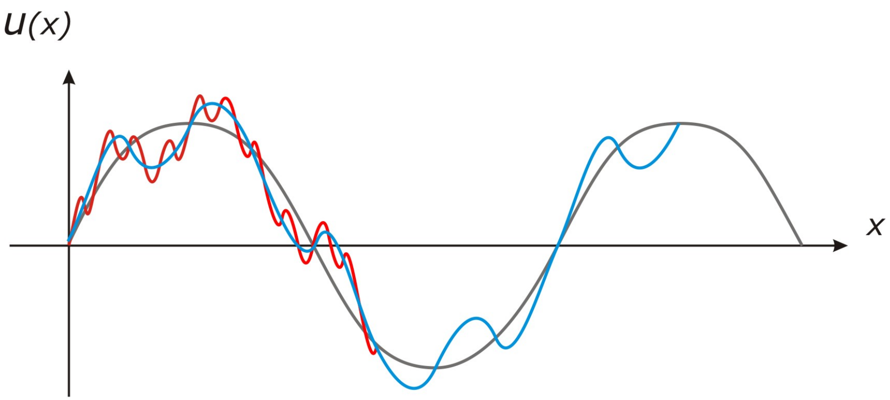
Lagrangian trajectory (L.Biferale et al)

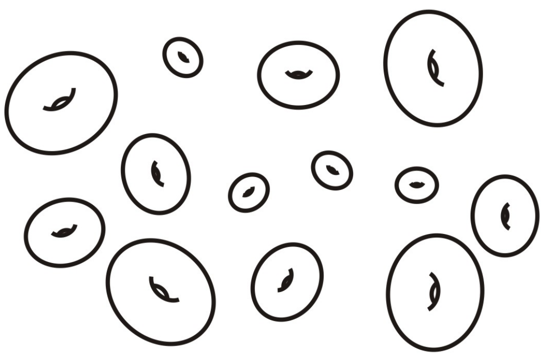


Cascade versus Singularity

- Cascade

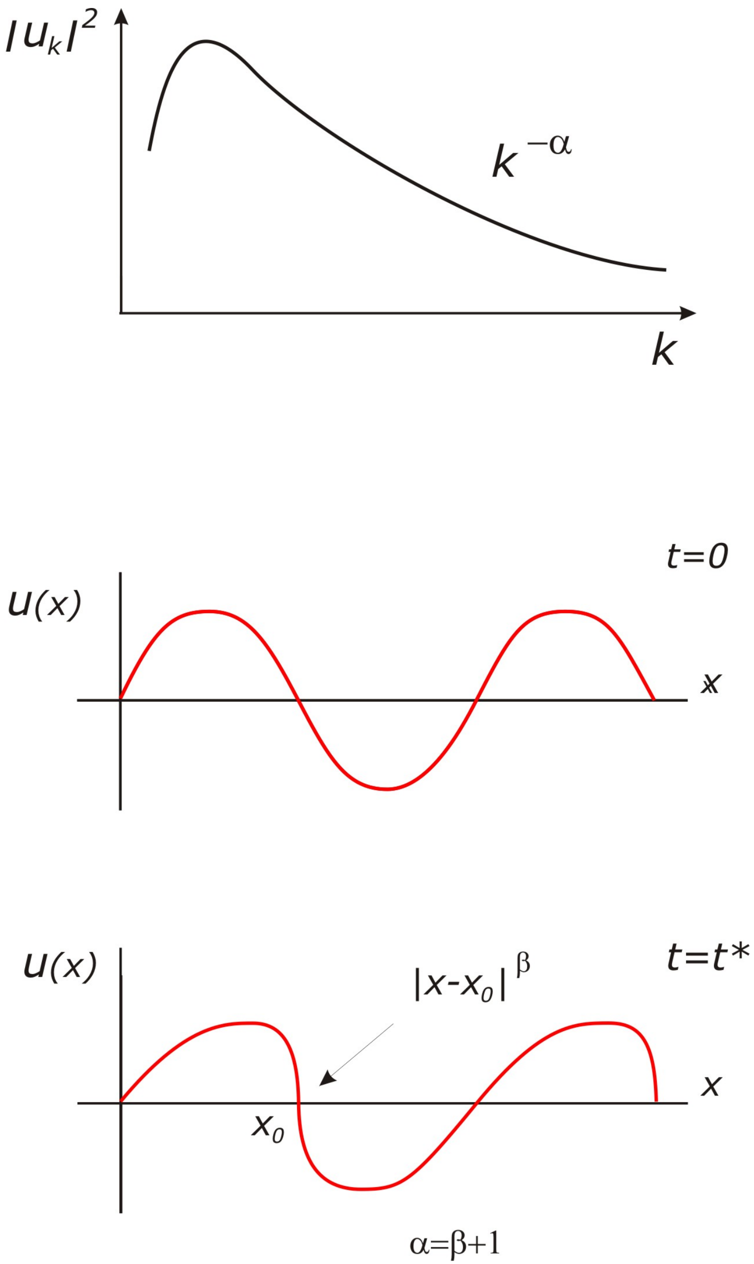






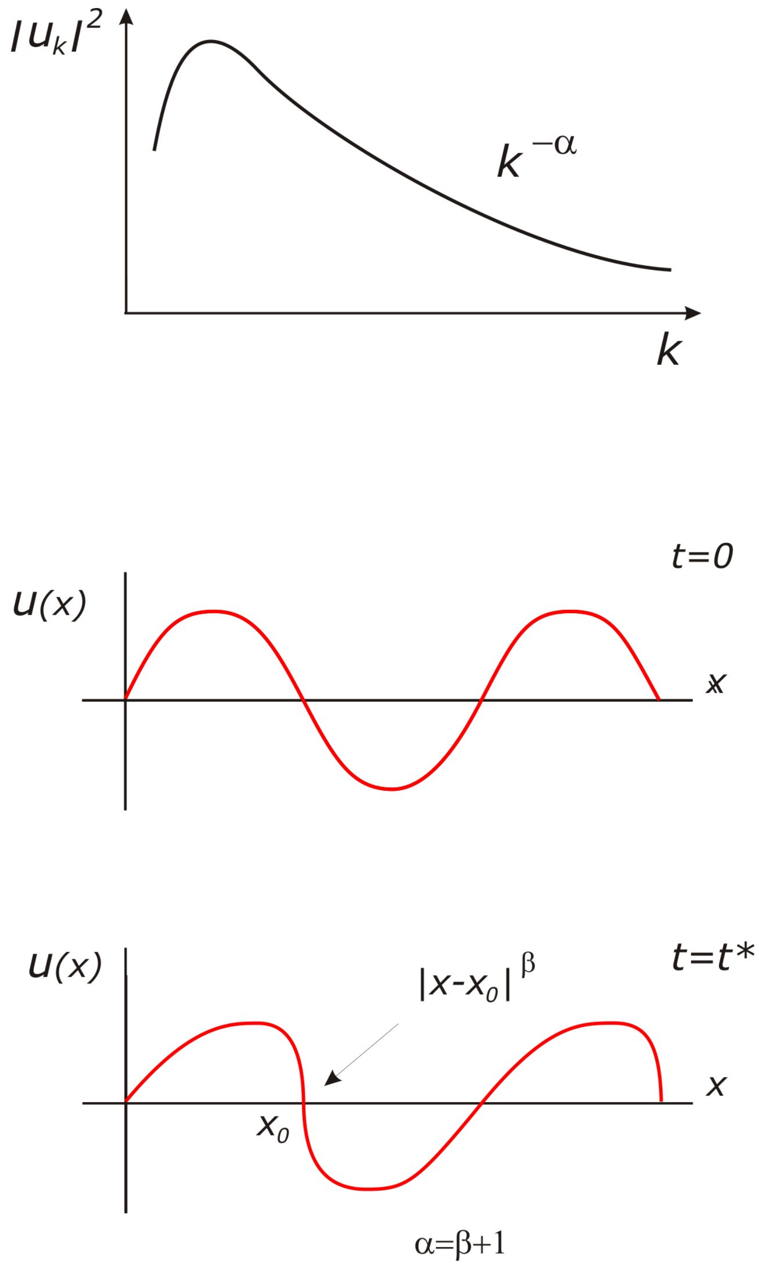
Cascade versus Singularity

- Singularity

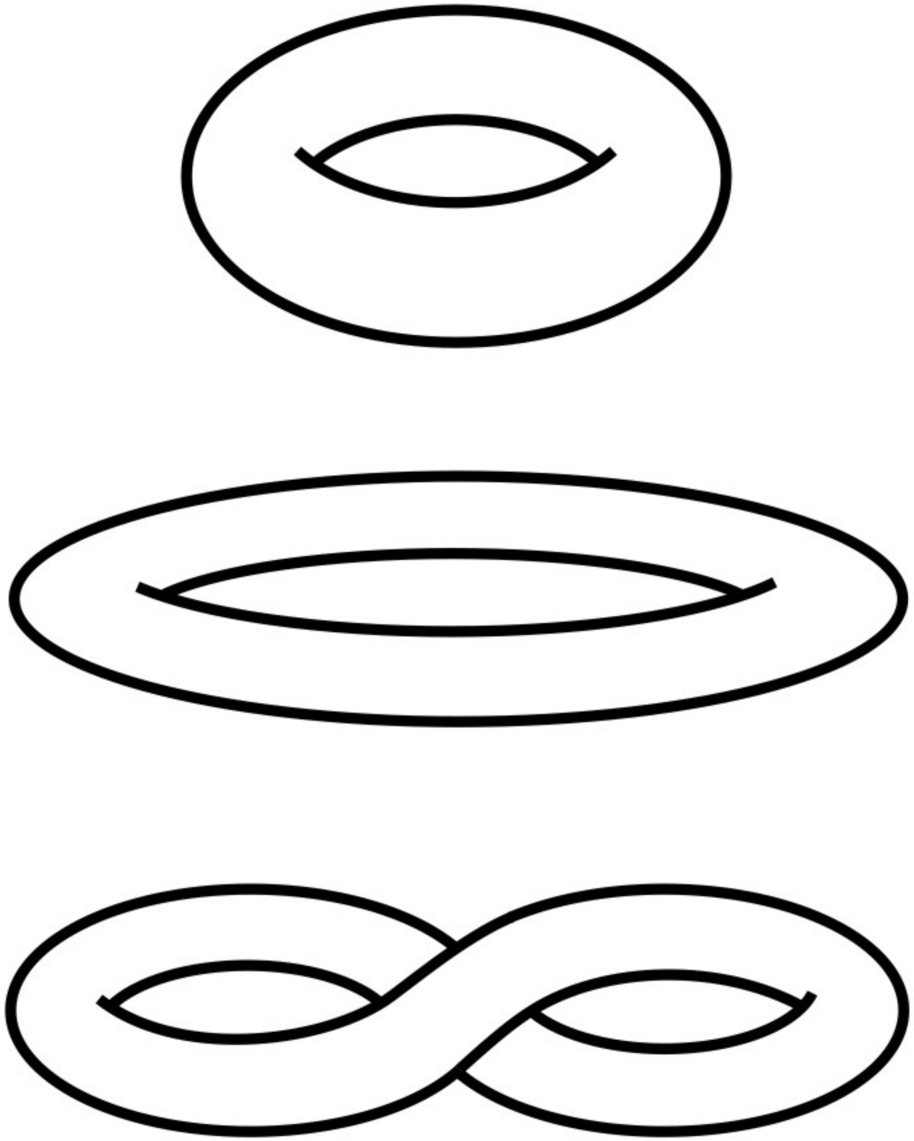


Cascade versus Singularity

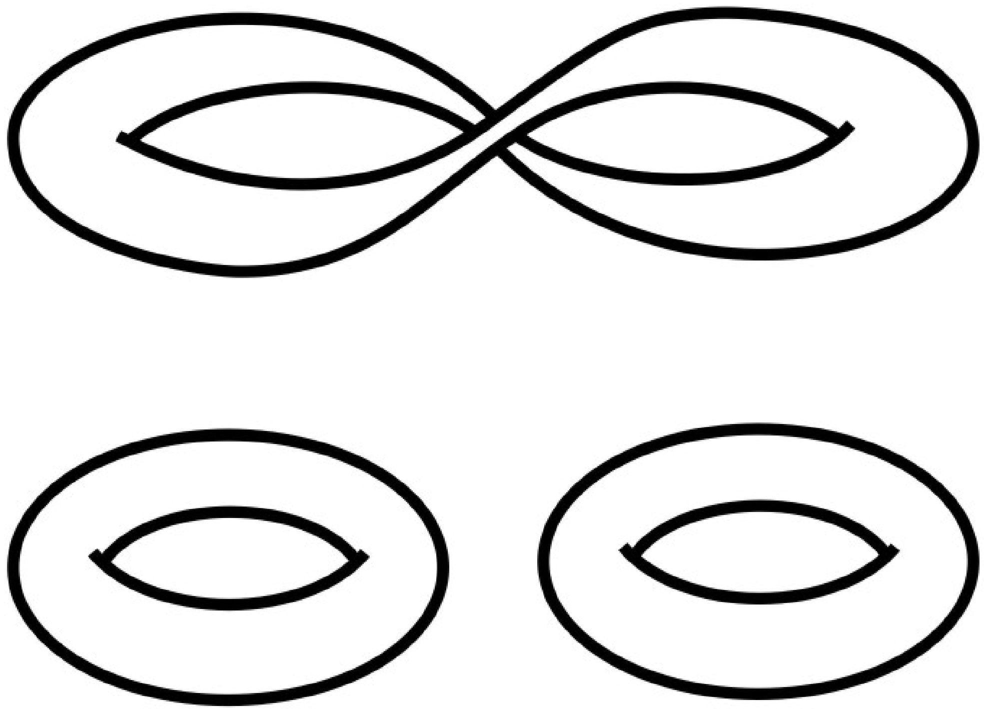
- Singularity



Cascade



Cascade is impossible without singularity



Multifractal model (*Parisi Frisch 1985*)

- The model generalizes the Kolmogorov theory (K41) to describe the observed nonlinear dependence of scaling exponents on their order
- Euler equations are invariant under the transformations

$$r \rightarrow r' = \gamma r \ , \quad v \rightarrow v' = \gamma^h v \ , \quad t \rightarrow t' = \gamma^{1-h} t$$

- assumption: determinative contribution to velocity structure functions is given by $\delta v(l) \sim l^h$ (spectrum of singularities?!)

$$\langle \Delta v^n \rangle = \int l^{nh} l^{3-D(h)} d\mu(h)$$

- The introduction of "fractal dimension" $D(h)$ follows naturally from the theory of large deviations

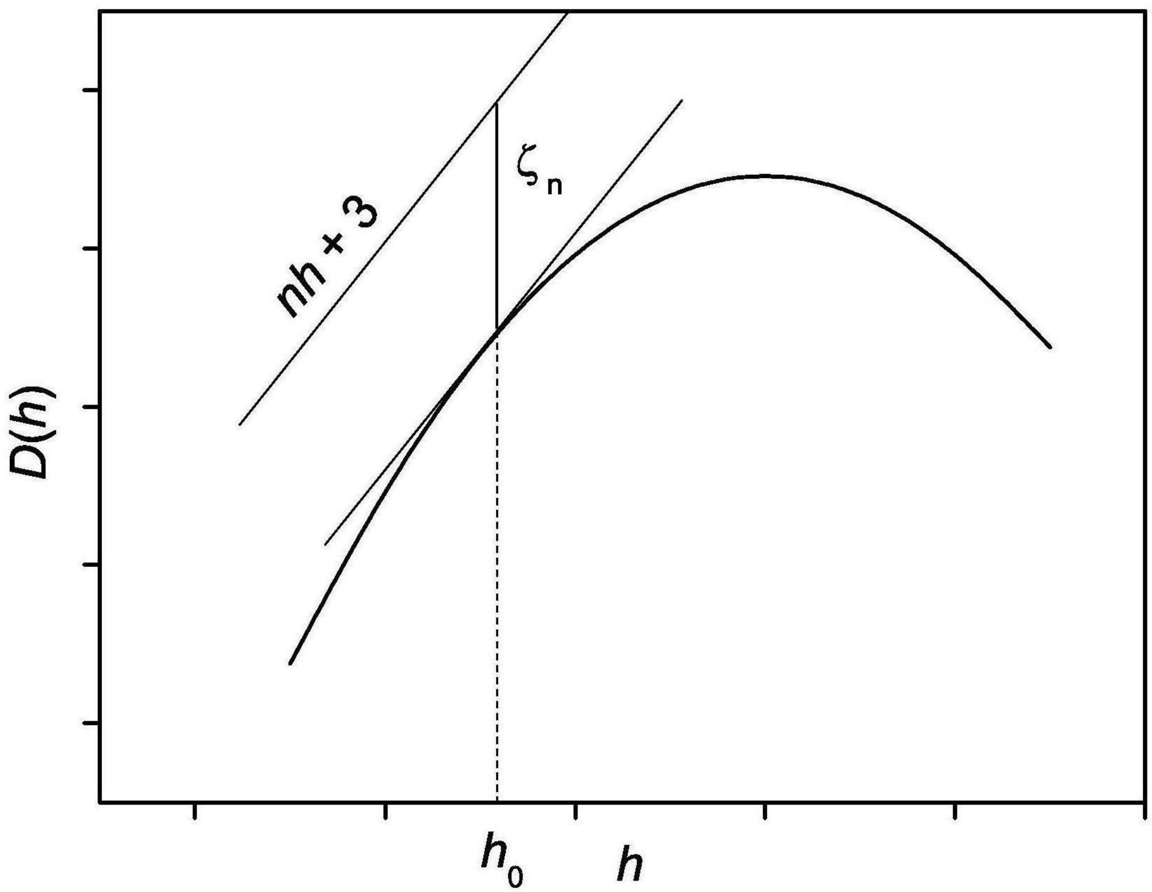
$$D_{\|} \ , \quad D_{\perp} \ ?$$

Multifractal theory

- In the limit $l \to 0$, only the smallest exponent contributes to the integral

$$\zeta_n = \min_h \left(nh + 3 - D(h) \right) , \qquad \lim_{l \to 0} \frac{\ln \langle \Delta v^n \rangle (l)}{\ln l} = \zeta_n ,$$

- ζ_n relates to $D(h)$ by the Legendre transformation



The statement of the problem

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla P + \mathbf{F}(r,t) + \nu\Delta\mathbf{v}\,, \qquad \nabla\cdot\mathbf{v} = 0$$

Introduction of randomness

Let $U_i(r,t)$ – some large-scale random velocity field

$$U_i(r,t) = \frac{1}{L^3}\int Q_i(\mathbf{r}+\rho,t)e^{-\rho^2/L^2}d\rho\,, \qquad \nabla\cdot\mathbf{U} = 0$$

Now we define large-scale stochastic force $F(r,t)$ by relation

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\mathbf{U} = -\nabla\pi + \mathbf{F}(r,t) + \nu\Delta\mathbf{U}\,, \qquad \nabla\cdot\mathbf{F} = 0$$

We substitute **F** on the right-hand side of NS equation. Seek the solution in the form

$$\mathbf{v}(\mathbf{r},t) = \mathbf{U} + \mathbf{u}(\mathbf{r},t)\,, \qquad P = p + \pi$$

$$\frac{\partial u_i}{\partial t} + (\mathbf{U}\nabla)u_i + (\mathbf{u}\nabla)U_i + (\mathbf{u}\nabla)u_i = -\nabla_i p\,, \qquad \nabla\cdot\mathbf{u} = 0$$

- This smoothed functions can be expanded in Taylor series for $r \ll L$

$$U_i(r,t) = U_i(0,t) + A_{ij}(t)r_j + A_{ijk}\frac{r_j r_k}{L}\ldots\ , \qquad A_{ii} = 0$$

- In the limit $L \to \infty$ but turnover time $T = const$ only first two terms remain
- the velocity $U_i(0,t)$ can easily be taken zero by choosing the reference frame
- NS equation takes the form

$$\frac{\partial u_i}{\partial t} + (A_{jk} r_k \nabla_j)u_i + A_{ik} u_k + (\mathbf{u}\nabla)u_i = -\nabla_i p\,, \qquad \nabla\cdot\mathbf{u} = 0$$

Velocity fluctuations

let $A_{ij}(t)$ be a random function of time

$$v_i = A_{ij}(t) r_j + u_i(r,t)\,, \qquad P = p + \nabla_i \nabla_j P(0,t) r_i r_j$$

$u_i(r,t)$ – velocity pulsation

$$\frac{\partial}{\partial t} u_i + (A_{kj} r_j \nabla_k) u_i + A_{ik} u_k + (u\nabla) u_i = -\nabla_i p\,, \qquad \nabla_i u_i = 0$$

this is the main equation of our theory

Asymptotic analysis (inviscid limit)

let $\quad u_i(r,t) = g_{i\mu}(t) w_\mu(X_\nu, t)\,, \qquad X_\nu = q_{\nu\alpha}(t) r_\alpha$

where $g_{i\mu}(t)$ and $q_{\nu\alpha}(t)$ satisfy the equations:

$$\dot{g}_{i\alpha} + A_{ij} g_{j\alpha} = 0\,, \qquad g_{i\alpha}(0) = \delta_{ij}$$

$$\dot{q}_{\gamma\nu} + q_{\gamma\mu} A_{\mu\nu} = 0\,, \qquad q_{ij}(0) = \delta_{ij}$$

let $A = A^T$, hence $g_{ij} = q_{ji}$ The equation then becomes

$$\frac{\partial w_\mu}{\partial t} + q_{\kappa\gamma} g_{\gamma\alpha} w_\alpha \frac{\partial w_\mu}{\partial X_\kappa} = -\frac{\partial P}{\partial X_\mu}\,, \qquad q_{\nu i} g_{i\mu} \frac{\partial w_\mu}{\partial X_\nu} = 0$$

Asymptotic behavior of q, g

- discrete approximation ($A = A^T$ is not required)
  let $A_{ij}(t) = (A_n)_{ij}$ be constant inside each small (n-th) interval

$$q_N = e^{-A_1 \Delta t} \cdot e^{-A_2 \Delta t} \cdot \ldots \cdot e^{-A_N \Delta t}$$

- production of $N \to \infty$ unimodular matrixes

The Theorem

Furstenberg, Tutubalin, Molchanov, Nechaev, Sinai ... see review Letchikov, UMN, v51, vypusk 1(307), 1996

- Iwasawa decomposition of the matrix $q = z(q)d(q)s(q)$ z is an upper triangular matrix with diagonal elements equal to 1, d is a diagonal matrix with positive eigenvalues, s is an orthogonal matrix

$$z(q_N) \to z_\infty$$

$$d(q_N) = \mathrm{diag}\left(e^{\lambda_1 N + O_1(\sqrt{N})}, e^{\lambda_2 N + O_2(\sqrt{N})}, e^{\lambda_3 N + O_3(\sqrt{N})}\right) ,$$

$$\lambda_1 < \lambda_2 < \lambda_3 , \qquad O_1 , O_2 , O_3 \qquad \text{Gaussian noise}$$

Simplifications

- there is a strong exponential growth

$$(qg)_N = (qq^T)_N \simeq z_\infty d(q_N) z_\infty^T$$

$$d(q_N) = e^{2\lambda_3 N} \cdot \mathrm{diag}(0,0,1) + O(e^{\lambda_2 N})$$

- we neglect the terms growing slower than $e^{2\lambda_3 N}$
- introduce a new vector variable $\mathbf{V} = C\mathbf{w}$; C_{ij} is a constant matrix

$$\left(\mathbf{V}\frac{\partial}{\partial \mathbf{X}}\right)\mathbf{V} = -C\frac{\partial}{\partial \mathbf{X}}\Pi\,, \qquad \frac{\partial \mathbf{V}}{\partial \mathbf{X}} = 0\,, \qquad P = e^{2\lambda_3 t}\Pi$$

  Stationary equation without randomness. This is due to the chosen variables $(\mathbf{X}, \mathbf{V})$; the randomness remains in rotation
- in reality nonlinearity depletion

$$\left(\mathbf{V}\frac{\partial}{\partial \mathbf{X}}\right)\mathbf{V} \approx 0$$

- recent papers suppot it (Gibbon et al 2014),(Kuznetsov 2015)

Analysis of the solution

- To understand the properties of the solution , we have to rewrite it back in laboratory coordinates $(\mathbf{r}, \mathbf{u})$.
- To separate the stochastic rotational part of the solution, we make one more change of variables

$$\mathbf{r}' = s\mathbf{r} \ , \qquad \mathbf{u}' = s\mathbf{u}$$

  after some manipulations

$$u_i' = e^{\lambda_i t} V_i(e^{\lambda_1 t} r_1', e^{\lambda_2 t} r_2', e^{\lambda_3 t} r_3')$$

  (no summation is assumed)
- in the rotating coordinates $\mathbf{r}'$, the asymptotic solution is not random
- As $t \to \infty$, the third component u_3' dominates, and the solution stretches exponentially with different coefficients along different axes
- We now take the curl to find vorticity

$$\omega' \simeq \omega_1' = e^{-\lambda_1 t} f\left(e^{\lambda_3 t} r_3'\right)$$

- since $\omega' = s\omega$, the absolute values of vorticities are equal in the two frames, so $\omega = \omega'$

Analysis of the solution

- vorticity (and velocity) is transported from boundaries to the center
- in stationary conditions vorticity (and velocity) can't grow exponentially in a finite volume

$$< u^2 >= \frac{1}{V}\int_V u^2 d^3 r\Big|_{t\to\infty} = \sum_j \frac{1}{V}\int_{V_j} u^2 d^3 r > n \cdot const \cdot e^{\lambda_{min} t}$$

- Thus, in stationary conditions vorticity (and velocity) can grow exponentially in some points only
- we have to demand that at some boundary point (see below)

$$\omega(t, L) \sim 1$$

- With account of the boundary condition, $f(e^{\lambda_3 t'} L) \sim e^{\lambda_1 t'}$, for any t'; choosing t' as $e^{\lambda_3 t} r_3' = e^{\lambda_3 t'} L$

$$\omega(t, r_3') \propto \left(\frac{r_3'}{L}\right)^{\lambda_1/\lambda_3}$$

It is valid for $r_3' > L e^{\lambda_3 (t_0 - t)}$. At smaller r_3', the vorticity ω is determined by the initial condition

Simple model

- 'straighten' the random flow, excluding the matrix s (without rotation)
- Simplifications: fix diagonal A_{ij} and $u = u(x,t)$

$$v_x = a\,x\,, \qquad v_y = b\,y + u(x,t)\,, \qquad v_z = c\,z\,, \qquad a+b+c=0$$

One can get the exact equation for vorticity

$$\frac{\partial \omega}{\partial t} + a\,x\frac{\partial \omega}{\partial x} - c\,\omega = 0$$

- Let also $a < 0\,, \qquad b > 0\,, \qquad c = -(a+b) > b$
- the boundary condition $\omega(t,1) = 1$ The solution takes the form

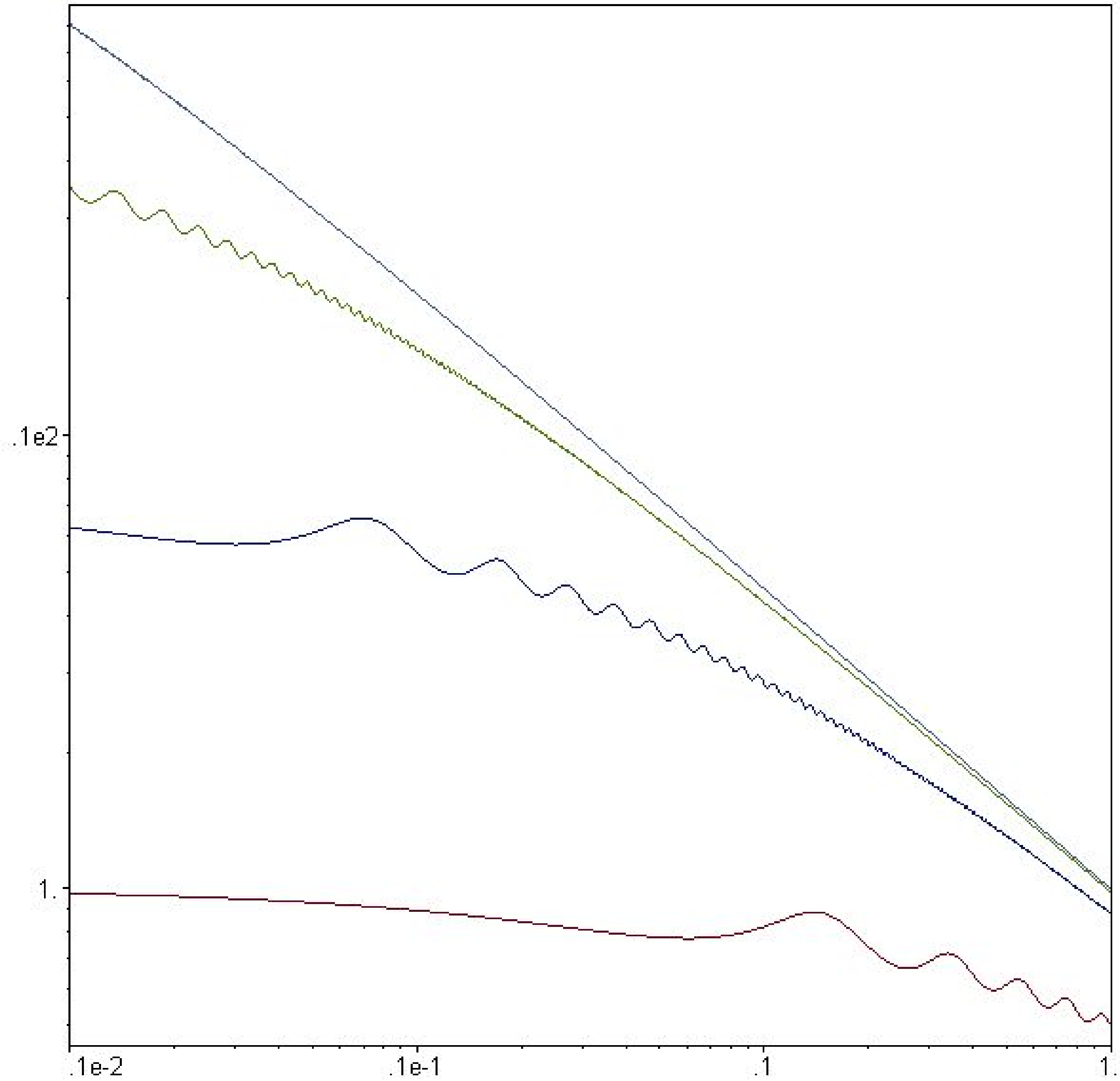
$$\omega(t,x) = e^{c(t-t')}\,\omega\left(t',1\right)\Big|_{t'(x)=t-(\ln x)/a} = x^{c/a}\,, \qquad x > \bar{x}(t) = e^{at}$$

$$\omega(t,x) = e^{ct}\omega_0\left(xe^{-at}\right)\,, \qquad x < \bar{x}(t)$$

- If the boundary condition is $\omega(t,1) = f(t)$

$$\omega(t,x) = x^{c/a} f\left(t - \frac{1}{a}\ln x\right) \to_{t\to\infty} x^{c/a} f(t)$$

Example of the solution



Evolution of spectrum

- The idea of cascade is based on power-law spectrum
- Let initial distribution of vorticity be

$$\omega_0(x) = (1 + ix)^{c/a} + (1 - ix)^{c/a}$$

The Fourier transform of this function is

$$\omega(k,t) = |k|^{b/a} e^{-|k| e^{at}} , \qquad a < 0$$

- The spectrum falls exponentially at $k \sim \bar{x}^{-1} = e^{-at}$
- Stationary fluctuations if $k \ll \bar{x}^{-1}$
  The result is similar to the effect of viscosity, but cutoff depends on time

Effect of viscosity

- It is easy to generalize and include the viscosity

$$\frac{\partial u(x,t)}{\partial t} + ax\frac{\partial u(x,t)}{\partial x} + bu(x,t) = \nu \frac{\partial^2 u}{\partial x^2}$$

- Changing to the variable $q = xe^{-at}$ we get

$$\frac{\partial \omega(q,t)}{\partial t} - c\omega(q,t) = \nu e^{-2at}\frac{\partial^2 \omega}{\partial q^2}$$

The Fourier transformation gives

$$\omega(k,t) = e^{-bt}\omega_0(ke^{at})e^{\frac{\nu}{2a}k^2(1-e^{2at})}$$

- For the example of initial condition considered in the previous slide

$$\omega(k,t) = |k|^{b/a}e^{-|k|e^{at}}e^{\frac{\nu}{2a}k^2(1-e^{2at})}, \qquad a < 0$$

Introduction of stochastics

- According to the Theorem, the stochastic generalization has the form

$$\frac{\partial \omega}{\partial t} + (a + \xi_1(t))x\frac{\partial \omega}{\partial x} - (c + \xi_2(t))\omega = 0$$

$\xi_1(t)$ and $\xi_2(t)$are Gaussian delta-correlated random processes

- The probability density

$$dP[\xi_1(t), \xi_2(t)] = e^{-\frac{1}{2D_1}\int \xi_1(t')^2 dt'} e^{-\frac{1}{2D_2}\int \xi_2(t')^2 dt'} \prod_t d\xi_1(t) d\xi_2(t)$$

the solution is

$$\omega(t, x) = e^{c(t-t') + \int_{t'}^{t} \xi_2(t'')dt''} \omega\left(t', xe^{-a(t-t') - \int_{t'}^{t} \xi_1(t'')dt''}\right)$$

- For $x = 0$,taking $t' = 0$, we get

$$\omega(t, 0) = e^{ct + \int_0^t \xi_2(t'')dt''} \omega(0, 0)$$

stochastic solution

- hence

$$\langle \omega(t,0)^n \rangle = e^{nct+n^2 D_2 t/2} \omega^n(0,0)$$

  This characterizes the solution inside the non-stationary inner region with growing vorticity

- $\bar{x}$ of the non-stationary region is determined by the condition

$$\bar{x} e^{-at-\int \xi_1 dt} \simeq 1$$

  But at $t \to \infty$: $\int \xi_1 dt \propto \sqrt{t}$ hence $\bar{x} \simeq e^{at}$ $\to$

$$\langle \omega^n \rangle = x^{nc/a} \int e^{\int \left(-\frac{\xi_2^2}{2D_2} + n\xi_2\right) dt} \prod_t d\xi_2(t) \omega^n(t',1) \propto x^{n\frac{c}{a} + n^2 \frac{D_2}{2a}}$$

- scaling of velocity moments is

$$\langle \Delta v^n(l) \rangle \sim \langle \omega^n \rangle l^n \sim l^{\zeta_n} \ , \quad \zeta_n = -\frac{b}{a} n + \frac{D_2}{2a} n^2$$

Discussion 1

- Average large-scale exponents λ_i determine the scaling (fractal) behavior of the solutions, while fluctuations of these exponents $\xi_1(t)\,,\xi_2(t)$ produce multifractality
- **Stretching of the vortex filaments is the main process**. Maximal stretching $(n \to \infty)$ is

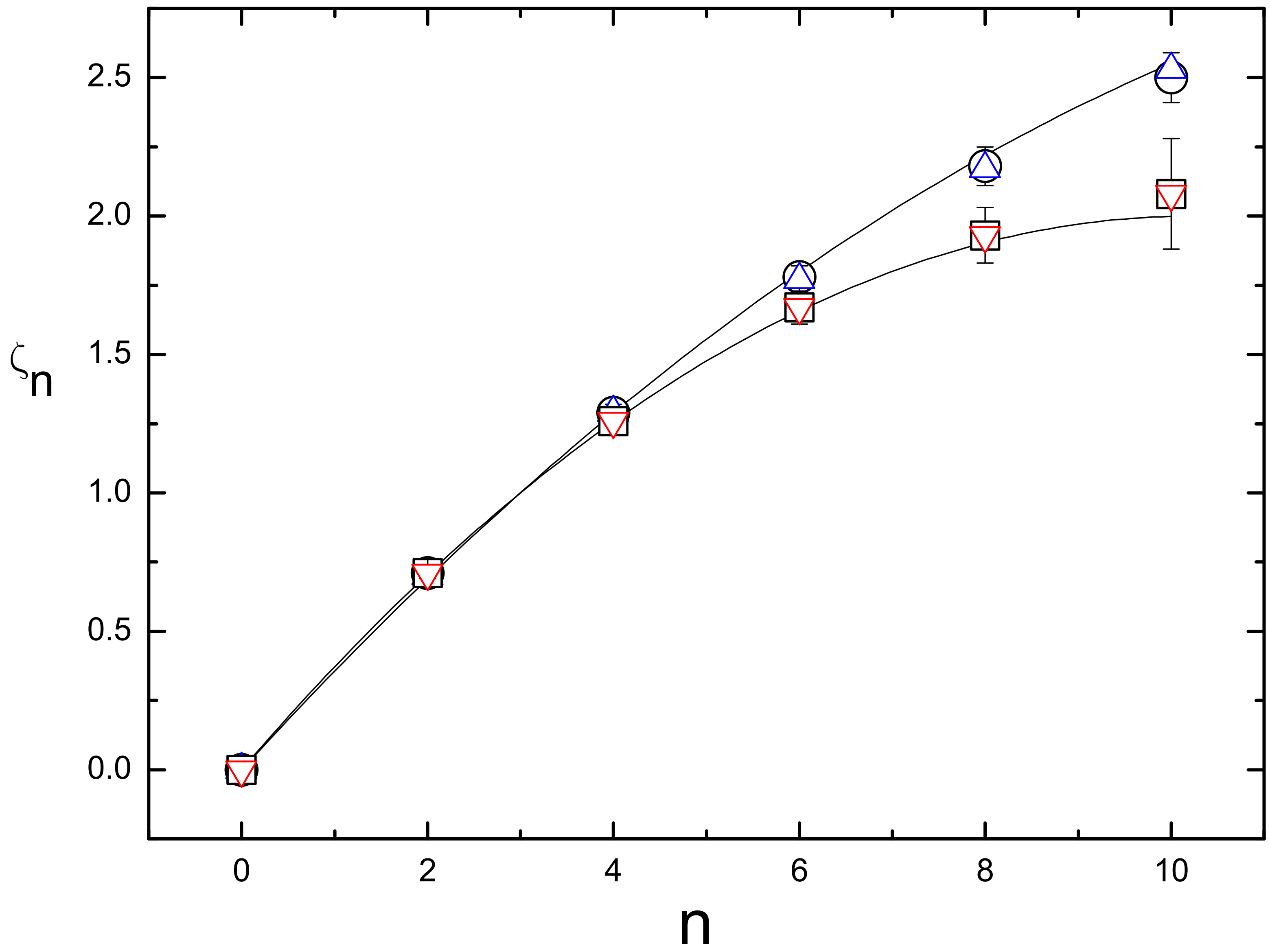
$$\mathbf{v} = \frac{[\mathbf{e}_z, \mathbf{r}]}{r}$$

- Structure functions

$$S_n^{\|} = 2\sqrt{\frac{2\pi}{n}}\frac{l^2}{en^2}\,, \qquad S_n^{\perp} = l^2\frac{2^n}{n}ln\frac{R}{l}$$

- At $n \to \infty$ there is a strong difference between $\|$ and $\perp$ exponents
- in simulations $\xi_{\|} > \xi_{\perp}$ longitudinal – sub-leading term !? $S_{\infty}^{\|} = 3$
- Taking into account $\xi_3 = 1$ one can get all structure functions

the result



Discussion 2

- **The main process is stretching of the vortex filaments, but not vortices breaking**
- If $P(A) = P(RAR^{-1})$ and $P(A_{ij}) = P(-A_{ij})$ the exponents are $\lambda_1 = -\lambda\,, \qquad \lambda_2 = 0\,, \qquad \lambda_3 = \lambda$
- $\lambda_2 = 0$ because the transformation $A \to -A$ is time reversal, but it is not true for turbulence – there is energy flux flowing into small scales

$$< \Phi >= \left\langle \int V^2 \mathbf{V} d\mathbf{s} \right\rangle \propto A_{ij} A_{jk} A_{ki} \propto det\, A$$

- Hence $\quad \lambda_2 \neq 0 \quad$ and $\qquad \lambda_1 < \lambda_2 < \lambda_3$
- Simple model $a < 0\,, \quad b > 0\,, \quad c > b$ corresponds to correct sign of energy flux

The assumptions and simplifications

- $r \ll L$ is not important, the approximation improves with time
- $\omega(1,t) = f(t)$ the structure function exponets do not change for any $f(t) < e^{\kappa t}$
- nonlinear dependence of structure function exponents on n are calculated for small D_2 only $\qquad (D_2 n/(2\,b) \ll 1)$
- depletion of nonlinearity $(v\nabla)v$ is obtained for the case $A^T = A$ in this case

$$q\,g\,(v\nabla)v = q\,q^T(v\nabla)v \propto e^{2\lambda_3 t} z_\infty diag\,(0,0,1)\,z_\infty^T(v\nabla)v$$

- if $A^T \neq A$ but $P(\Omega) = P(-\Omega)\,, \qquad 2\Omega = A - A^T$ in this case

$$q = z_{1\infty}\,d\,R_1(t)\,, \qquad g = R_2^{-1}(t)\,d\,z_{2\infty}^T$$

and nonlinearity

$$q\,g\,(v\nabla)v \propto e^{2\lambda_3 t} R_{33}(t)\,z_{1\infty} diag\,(0,0,1)\,z_{2\infty}^T(v\nabla)v\,,$$

rigorous analysis gives $\lambda_2(A) < 0$

- so, the result looks general

- **THUS:**
- We believe that $\xi_{\perp}^{n} < \xi_{\parallel}^{n}$ for some $n > N*$
  IN THIS CASE:
- $\xi_{\perp}^{n}$ is the leading asymptotic term $\xi_{\parallel}^{n}$ is sub-leading term
- It is very difficult to construct theory for sub-leading terms
  WE EXPECT:
- to calculate $\xi_{\perp}^{n}$, to get saturation and to find saturation level directly from NS equation.
- unsolved problem why λ_i are universal?