Intermittency and Universality in Fully Developed Inviscid and Weakly Compressible Turbulent Flows

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We perform high-resolution numerical simulations of homogenous and isotropic compressible turbulence, with an average 3D Mach number close to 0.3. We study the statistical properties of intermittency for velocity, density, and entropy. For the velocity field, which is the only quantity that can be compared to the isotropic incompressible case, we find no statistical differences in its behavior in the inertial range due either to the slight compressibility or to the different dissipative mechanism. For the density field, we find evidence of "frontlike" structures, although no shocks are produced by the simulation.

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Fully developed three-dimensional incompressible turbulence is characterized by an intermittent energy flux from large to small scales. According to the Kolmogorov theory, the statistical properties of turbulence are scale invariant within the inertial range $\eta \ll r \ll L_0$, where L_0 is the scale of energy forcing, and η the dissipative scale. Intermittency spoils dimensional scale invariance and is the origin of anomalous scaling [1]. Few investigations have been reported so far for the case of weakly compressible and inviscid turbulence, relevant to many astrophysical and geophysical problems [2,3]. In this Letter we present a high-resolution numerical simulation of homogeneous and isotropic, 3D compressible and inviscid turbulence. Our main purpose is to investigate intermittency in the inertial range and compare our finding against known results for incompressible viscous turbulence. For the case of driven and decaying supersonic turbulence similar problems have been addressed in [4]. For this simulation, we use the FLASH 3 component-based simulation framework. While the FLASH framework was primarily designed to treat compressible, reactive flows found in astrophysical environments [5], it is generally applicable to many other types of fluid phenomena. For this simulation, only the compressible hydrodynamics module based on the higher-order Godunov piecewise parabolic method (PPM) was used [6]. The algorithmic methodology of the FLASH framework have been described in further detail elsewhere [7]. The equation of motions solved are the Euler equations for a fluid of density ρ with forcing, **F**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\boldsymbol{v}\rho) = 0, \tag{1}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{v}\rho) = -\nabla P + \boldsymbol{F}, \tag{2}$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [\boldsymbol{v}(\rho E + P)] = 0, \tag{3}$$

with the equation of state: $P = (\gamma - 1)\rho U$ and $E = U + 1/2\rho v^2$ and where P is the pressure, E the total energy, U the internal energy, and γ is the ratio of the specific heats in the system. The effect of the large scale forcing $\boldsymbol{F}$ in (2) gives rise to a turbulent flow whose energy is transferred from scale L_0 towards small scales [7]. The energy input $\int \boldsymbol{v}\boldsymbol{F} d^3x$ produces an increase of the internal energy U, which grows in time. There are two mechanisms of kinetic energy dissipation. The most important is the energy transfer from kinetic to heat via the compressible effects; the second is a numerical smoothing of steep velocity and density gradients tuned to filter out local numerical instability. The latter is important only at scales of the order of the grid spacing [7]. The quantity $\int dx^3 P \partial_i v_i$ represents the energy transfer from kinetic to internal energy of the flow. The mean sound speed increases slightly in time as well, though the 3D rms Mach number is roughly 0.3 (1D Mach number 0.17) throughout. The numerical simulation was done for isotropic and homogeneous forcing [8] with a resolution of 1856^3 grid points. The integration in time was done for 3 eddy turnover times after an initial transient evolution beginning from rest. The numerical treatment of turbulent flows used in the FLASH simulation is sometimes referred to as an implicit large eddy simulation (ILES), to be distinguished from a full direct numerical simulation (DNS) of the Navier-Stokes (NS) equations. Thus, one might expect that the dynamics of the flow may differ from incompressible and viscous high

Reynolds number turbulence. A key question is therefore whether the inertial range statistics differ significantly with respect to the homogeneous and isotropic incompressible case. To answer this question, we measure the scaling properties of velocity fluctuations. In particular, we use the method of extended self-similarity (ESS) [9], which allows us to accurately estimate the anomalous exponents of the longitudinal $S_P^{(L)}(r) \equiv \langle[\delta \boldsymbol{v}(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}}]^p\rangle$ (where $\delta\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})$ and the average is over the volume and in time) and transverse structure functions $S_P^{(T)}(r) \equiv \langle[\delta\boldsymbol{v}(\boldsymbol{r}_T)]^p\rangle$ (where $\boldsymbol{r}_T \cdot \boldsymbol{v} = 0$). We denote the corresponding scaling exponents by $\zeta_p(L)$ and $\zeta_p(T)$. Our numerical result for $\zeta_p(L)$, $\zeta_p(T)$ agrees remarkably well with previous data. This is shown in Fig. 1, which compares the ESS local slope $d\log(S_P(r))/d\log(S_2(r))$, for both longitudinal and transverse structure functions $p = 4$ and $p = 8$ with the DNS simulations performed for incompressible NS equation at comparable Reynolds numbers [10]. Figure 1 shows several interesting features. First, there exists a range of scales $(r/\eta \geq 50)$ where the local slope is almost constant, i.e., where we can detect accurately an anomalous scaling exponent. Second, as one can see, in the inertial range our ILES results give exactly the same anomalous scaling as that obtained for the incompressible NS equation [10]. There is a clear difference between our results and the NS case in the dissipation range $(r/\eta \leq 20)$, where the NS solutions show a well-defined "dip" effect (as qualitatively predicted by the multifractal theory [11]), while ILES behaves differently. The different behavior in the dissipation range is expected since the ILES does not dissipate energy in the standard way. On the other hand, the remarkable agreement in the inertial range allows us to claim that the inertial range properties are independent of the dissipation mechanism [12]. This supports the conjecture that the statistical properties of turbulence in the inertial range are universal and independent of the dissipation mechanisms. This is one of our main results.

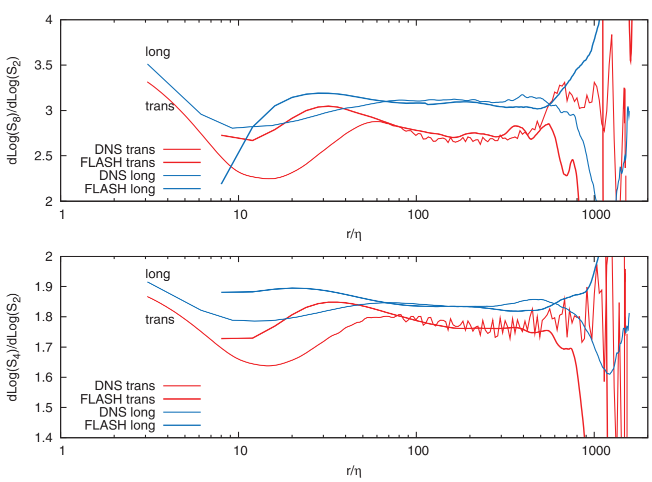

FIG. 1 (color). Local slopes of the longitudinal and transverse structure functions from the present ILES (thick lines) and compared against a DNS [10] at comparable Reynolds (thin lines). The normalization with respect to η is subjected to a certain degree of arbitrariness, due to the fact that we can estimate it only indirectly (see text).

We also note that the difference between longitudinal and transverse scaling exponents observed in homogeneous and isotropic DNS [10], is also seen in our numerical results. This discrepancy is an open theoretical issue, not explainable using standard symmetry argument in homogeneous and isotropic turbulence [13]. Whether this remains true at higher Reynolds numbers is an open question [14].

Although the integration is formally inviscid, there is a net energy transfer from the turbulent kinetic energy $1/2\rho v^2$ to the internal energy. Thus we may consider that an effective viscosity $\nu_{\rm eff}$ is acting on the system. In order to estimate $\nu_{\rm eff}$ we proceed as if the Kolmogorov equation—with effective parameters—applies to our case: $S_3^{(L)}(r) = -\frac{4}{5}\epsilon_{\rm eff} r + 6\nu_{\rm eff}\frac{d}{dr}S_2^{(L)}(r)$ A fit of our data with this formula gives, $\epsilon_{\rm eff} = 0.054$, $\nu_{\rm eff} = 8.3 \times 10^{-6}$, which corresponds to a Kolmogorov scale, $\eta = (\nu_{\rm eff}^3/\epsilon_{\rm eff}^{1/4})$, equivalent to roughly half grid cell and to $R_\lambda \sim 600$. The amount of kinetic energy transferred to internal energy by the term $\langle \boldsymbol{\partial} \cdot \mathbf{v}P\rangle$ is around 60%–70% of the energy input as given by the estimate of $\epsilon_{\rm eff}$ in the previous expression. The dynamical effects of the effective viscosity are however different from what one usually observes in the NS equations; i.e., the dissipation range does not behave the same as in the NS solutions. In Fig. 2 we show the density field in the system at a given time. One can easily recognize the existence of large density gradients due to compressibility. Another interesting quantity to look at is the entropy S defined as $S \equiv \log(P/\rho^\gamma)$. Using Eqs. (1)–(3) one obtain

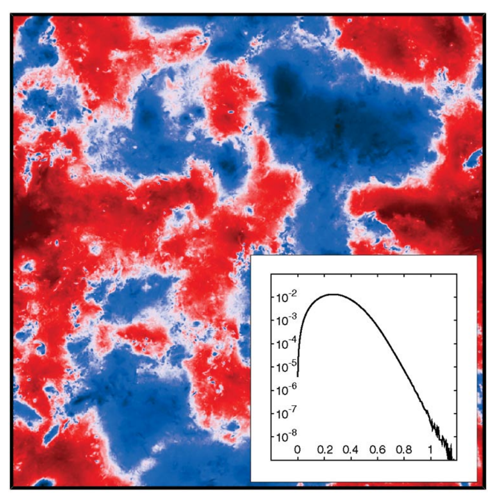

FIG. 2 (color). 2D section of the density field ρ at a given time: large regions with smooth density variations are separated by sharp cliffs. Inset: probability density function of Mach number the following equation for the entropy: $\partial_t S + \boldsymbol{v} \cdot \boldsymbol{\nabla} S = 0$. This equation tells us that S satisfies an evolution similar to the case of a passive scalar advected by the velocity vector $\boldsymbol{v}$, although S cannot be considered passive here. Strong variations are also detectable in the field S (not shown). Recently, the statistical properties of density fluctuations have been investigated for supersonic turbulence characterized by large Mach number and a rather large effect is due to the formation of shock waves and fronts [4,15]. In our case, the 3D Mach number is of order 0.3 on average and, consequently, we should expect that shock waves are not important. In the inset of Fig. 2, we show the whole probability density function (PDF) of Mach number distribution for a given time during the temporal evolution. As shown, the compressibility degree may reach a maximum excursion where Mach $\sim O(1)$, indicating that compressible effects cannot be however fully neglected in the flow. Actually, it has been shown in experiments and direct numerical simulations of scalar quantities, that frontlike structures are frequently observed [16,17]. A frontlike structure on a quantity $\mathcal{O}$ is characterized by a "local scaling" property $\mathcal{O}(x + r) - \mathcal{O}(x) \sim$ const for $x + r$ and x selected on the two different sides of the front. If these "frontlike" structures play a significant role in the statistical fluctuations of $\mathcal{O}$, one should observe intermittent anomalous scaling for the structure functions of $\mathcal{O}$. In particular the anomalous exponents should approach a constant value for $p \to \infty$.

In order to study the statistical properties of the density and entropy fluctuations, we can introduce the density structure functions $D_p(r) = \langle[\delta\rho(r)]^p\rangle$ and the entropy structure functions $E_p(r) = \langle[\delta S(r)]^p\rangle$. The ILES result shows that $D_P(r)$ and $E_p(r)$ are scaling functions of r in the inertial range: i.e., $D_p(r) \sim r^{z_p}$ and $E_p(r) \sim r^{h_p}$. The values of z_p and h_p are shown in the insert of Fig. 3 together with the compensated plot $D_5 r^{-z_5}$ and $D_6 r^{-z_6}$. From Fig. 3 one can appreciate the quality of the scaling in the inertial range. We remark that the scaling exponents z_p and h_p show quite anomalous behavior. For $p \leq 3$ the values of z_p are larger than the corresponding values of $\zeta_p(L)$ and $\zeta_p(T)$ which means that density is in average somehow smoother than the velocity field. The anomalous exponents for both the density and the entropy structure functions become constant at large order. In particular, defining the saturation exponents to be z_∞ and h_∞, we estimate $z_\infty = 1.62 \pm 0.10$ and $h_\infty = 1.00 \pm 0.10$. A similar analysis for the pressure field (not shown) shows that the structure functions of the pressure field P have the same scaling exponents and of the same saturation exponents as those for the density field.

In Fig. 4 we show the statistics of entropy and density fluxes, defined as $\Pi_\rho(r) = \delta\rho(r)^2(\delta\boldsymbol{v}(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}})$; $\Pi_S(r) = \delta S(r)^2(\delta\boldsymbol{v}(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}})$. In particular, in Fig. 4, panel (a) and (b); we show the scaling exponents $R(p)$, $Q(p)$ defined as $\langle|\Pi_\rho|^{p/3}(r)\rangle \sim r^{R(p)}$ and $\langle|\Pi_S|^{p/3}(r)\rangle \sim r^{Q(p)}$, where we

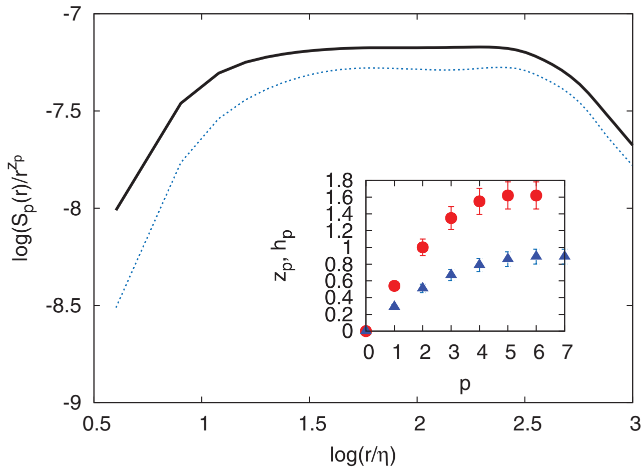

FIG. 3 (color). Compensated structure functions of the density $D_6(r)r^{-z_\infty}$ (solid line) and $D_5(r)r^{-z_\infty}$ (dashed line). Inset: scaling exponents of the density (●) and entropy (▲) structure functions. Errors include both statistical fluctuations and the uncertainty in the fit to the inertial range.

have used the absolute value to improve the statistical convergence. Both exponents are anomalous, and none of them show the signature of a constant transfer of fluctuations toward small scales, i.e., $R(3) \neq 1$; $Q(3) \neq 1$. This implies that the phenomenology of density and entropy transfer are significantly different from what observed for passive scalar quantities in incompressible

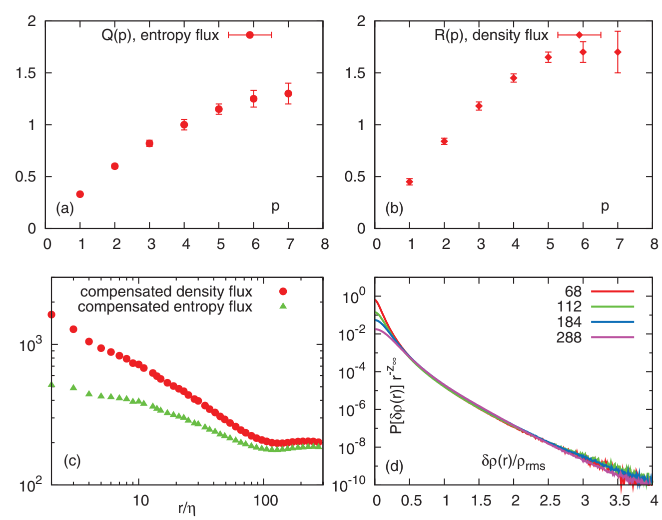

FIG. 4 (color). (a) scaling exponents of entropy flux, $\langle|\Pi_S|^{p/3}\rangle$; (b) the same of panel (a) but for density flux $\langle|\Pi_\rho|^{p/3}\rangle$; (c) sixth order moment of density flux compensated with velocity and density fluctuations, $\langle\Pi_\rho(r)^2\rangle/(S_2^L(r)D_4(r))$ (circle) and the same quantity but for entropy flux $\langle\Pi_S(r)^2\rangle/(S_2^L(r)E_4(r))$ (diamonds); (d) Rescaled probability distribution function, $P[\delta\rho(r)]r^{-z_\infty}$. Different curves corresponds to different values of $r/\eta = 68$, 112, 184, 288 in the inertial range.

turbulence. In order to understand the correlation between these two quantities and the advecting velocity field, we show in panel (c) of Fig. 4 the compensated plot: $\langle \Pi_S^2(r) \rangle / (S_2^L(r) E_4(r))$ and $\langle \Pi_\rho^2(r) \rangle / (S_2^L(r) D_4(r))$. Clearly, both density and entropy become more and more correlated with the velocity by going to small scales, with density always more correlated than entropy. This confirms the phenomenological idea that density fluctuations are strongly constrained by the fluctuations of $\partial \cdot v$ at small scales. The difference passive scalar advected by an incompressible velocity field is probably due to the presence of extra terms proportional to the correlation between density and velocity divergence in the equations governing the scalar fluctuations.

According to the multifractal theory, the effect of saturation in the exponents z_p and h_p is equivalent to saying that the tail of the probability distribution $P[\delta\rho(r)]$ should behave as $\sim r^{z_\infty}$ for any r. Thus we should expect that the functions $r^{-z_\infty} P[\delta\rho(r)]$ should collapse on the same distribution for all r in the inertial range [16]. In panel (d) of Fig. 4 we show that this is exactly the case for ILES result. Let us note that, as shown in Fig. 3, z_∞ is larger than h_∞. Using the multifractal theory, one can relate the saturation exponent to the fractal dimension of the frontlike structures: i.e., $D_\rho = 3 - z_\infty$ and $D_S = 3 - h_\infty$. Our findings show that D_S is larger than D_ρ: i.e., fronts in the entropy field are easier to form compared to those in the density field. This result may not be surprising if we observe that large entropy fluctuations are produced by large pressure *or* density fluctuations. The above argument implies that entropy is a more intermittent quantity than density and pressure. Entropy is a conserved quantity along Lagrangian trajectories. This could suggest some connection between the existence of frontlike structures and the behavior of inertial particles in *incompressible* turbulence, where it is known that particles tend to form multifractal sets with correlation dimension as low as 2 [18,19].

Let us summarize our results. Using a numerical simulation of inviscid homogeneous, isotropic weakly-compressible turbulence, we find that the scaling properties of the velocity field in the inertial range are in excellent agreement with those observed in DNS of the NS equations [10]. This result supports the statement that the nature of the dissipation does not affect the statistical properties of the inertial range; i.e., turbulence is universal with respect to the dissipation mechanism. We confirm that transverse and longitudinal structure functions show different scaling properties (up to this Reynolds number). We have also shown that, although almost no shock waves are produced in the simulation, the density fluctuations are characterized by frontlike structures. Accordingly, the scaling exponents of density and entropy structure functions, D_p and E_p saturate at large p.

The presence of frontlike structures as in passive scalars advected by incompressible flows [16,17], is not necessarily a signature of a passivelike behavior, as for instance shown by the fluctuations of the velocity field in Burgers' equations. Universality of the density and entropy fluctuations at changing the Mach number, is not expected. In particular, for supersonic flows, the probability to observe fronts (or shocks) becomes larger and more correlated with the velocity fluctuations. Therefore, exponents may saturate at different values for large orders, as seen for the scaling properties of the entropy in [4].

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