On the role of helicity for large- and small-scales turbulent fluctuations

Ganapati Sahoo,[1] Fabio Bonaccorso,[1] and Luca Biferale[1]
[1] *Department of Physics & INFN, University of Rome Tor Vergata, Via della Ricerca Scientifica 1, 00133 Rome, Italy.*
(Dated: June 17, 2015)

The effect of the helicity on the dynamics of the turbulent flows is investigated. The aim is to disentangle the role of helicity in fixing the direction, the intensity and the fluctuations of the energy transfer across the inertial range of scales. We introduce an external parameter, α, that controls the mismatch between the number of positive and negative helically polarized Fourier modes. We present the first set of direct numerical simulations of Navier-Stokes equations from the fully symmetrical case, $\alpha = 0$, to the fully asymmetrical case, $\alpha = 1$, when only helical modes of one sign survive. We found a singular dependency of the direction of the energy cascade on α, measuring a positive forward flux as soon as only a few modes with different helical polarities are present. On the other hand, small-scales fluctuations are sensitive only to the degree of mode-reduction, leading to a vanishing intermittency already for values of $\alpha \sim 0.1$ and independently of the degree of mirror symmetry-breaking. Our findings suggest that intermittency is the result of a global mode-coupling in Fourier space.

arXiv:1506.04906v1 [nlin.CD] 16 Jun 2015

The direction of the energy transfer in a turbulent flow is believed to be determined by the combined effects of all inviscid invariants which depends on the embedding dimensionality and/or on the coupling with external fields as in conducting or buoyant systems [1–3]. Among the inviscid invariants, those have a definite sign are key, e.g., for fully homogeneous and isotropic turbulence (HIT) in two dimensions the presence of two positive-definite invariants, energy and enstrophy, does not allow a stationary transfer of both quantities among the same window of scales, i.e., there exists a split cascade with energy flowing towards the large scales (inverse cascade) and enstrophy to the small scales [4–9]. The three-dimensional (3D) Navier-Stokes equations (NSE) possess two inviscid invariants, energy and helicity (the scalar product of velocity and vorticity) [10–12]). At a difference from the energy, helicity is not positive definite and it is observed to be preserved by some energy dissipative events such as anti-parallel vortex reconnection [13, 14]. As a result, it is not possible to predict the direction of the energy and helicity transfers. On one hand, numerical simulations, phenomenological arguments, dynamical models, closures and comparison with the inviscid Gibbs-like equilibrium distribution suggest that both energy and helicity have a mean transfer to the small scales (direct cascade) in HIT [12, 15–21]. On the other hand, it is well known that the external mechanisms such as rotation [22, 23], confinement [24], shear [25] or coupling with the magnetic field [26] might revert the direction of the energy cascade due to local or non-local inverse energy transfer. Strikingly enough, such a reversal of the flux has been predicted and observed also in 3D HIT with explicit breaking of parity invariance, i.e., by restricting the dynamics to a subset of Fourier modes such that the helicity becomes sign definite [27–29], suggesting that inverse energy transfer events are much broader than previously thought and they are potentially present in all flows in nature. In this letter we address further the latter observation by systematically investigating the effects of the helical mode-reduction in 3D NSE; the aim is to explain the role played by the helicity in fixing the direction, the intensity, and the fluctuations of the mean energy flux.

The key new tool is based on a suitable projection of the NSE allowing to disentangle, *triad-by-triad*, the properties of the energy transfer as a function of the percentage of negative helically polarized modes kept in the simulation. The existence of a control parameter is crucial to address the problem in a quantitative way, tailoring the degrees of freedom kept and removed, without any modeling. We start with the helical decomposition [16] of the velocity field $\boldsymbol{u}(\boldsymbol{x})$, expanded in Fourier series $\boldsymbol{u}_{\boldsymbol{k}}$, as

$$\boldsymbol{u}_{\boldsymbol{k}} = u^+_{\boldsymbol{k}} \boldsymbol{h}^+_{\boldsymbol{k}} + u^-_{\boldsymbol{k}} \boldsymbol{h}^-_{\boldsymbol{k}}, \quad (1)$$

where $\boldsymbol{h}^\pm_{\boldsymbol{k}}$ are the eigenvectors of the curl, i.e., $i\boldsymbol{k} \times \boldsymbol{h}^\pm_{\boldsymbol{k}} = \pm k \boldsymbol{h}^\pm_{\boldsymbol{k}}$. We choose $\boldsymbol{h}^\pm_{\boldsymbol{k}} = \hat{\nu}_{\boldsymbol{k}} \times \hat{k} \pm i\hat{\nu}_{\boldsymbol{k}}$, where $\hat{\nu}_{\boldsymbol{k}}$ is an unit vector orthogonal to $\boldsymbol{k}$ satisfying the condition $\hat{\nu}_{\boldsymbol{k}} = -\hat{\nu}_{-\boldsymbol{k}}$, e.g., $\hat{\nu}_{\boldsymbol{k}} = \boldsymbol{z} \times \boldsymbol{k} / ||\boldsymbol{z} \times \boldsymbol{k}||$, with any arbitrary vector $\boldsymbol{z}$. In terms of such *exact* decomposition of each Fourier mode, the total energy, $E = \int d^3x\, |\boldsymbol{u}(\boldsymbol{x})|^2$, and the total helicity, $H = \int d^3x\, \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\omega}(\boldsymbol{x})$ are written as

$$E = \sum_{\boldsymbol{k}} |u^+_{\boldsymbol{k}}|^2 + |u^-_{\boldsymbol{k}}|^2; \quad H = \sum_{\boldsymbol{k}} k(|u^+_{\boldsymbol{k}}|^2 - |u^-_{\boldsymbol{k}}|^2), \quad (2)$$

where $\boldsymbol{\omega}$ is the vorticity. The nonlinear term of the NSE can be then decomposed in terms of the helical content of the complex amplitudes, $u^{s_{\boldsymbol{k}}}_{\boldsymbol{k}}$ with $s_{\boldsymbol{k}} = \pm$ (see [16]). In a triadic interaction within modes $u^{s_{\boldsymbol{k}}}_{\boldsymbol{k}}$, $u^{s_{\boldsymbol{p}}}_{\boldsymbol{p}}$, $u^{s_{\boldsymbol{q}}}_{\boldsymbol{q}}$, there exist eight possible helical combinations ($s_k = \pm$, $s_p = \pm$, $s_q = \pm$) falling into four independent classes because of the symmetry that allows simultaneous change of the sign of the helicity of each mode. We consider the dynamics of an incompressible flow ($\boldsymbol{\nabla} \cdot \mathbf{u} = 0$) determined by

RUN	N	k_f	α
RUN 1-8	256	$[1, 3]$	$0 - 0.999$
RUN 9-13	256	$[1, 3]$	$0.1 - 0.9$
RUN 14-19	512	$[1, 2]$	$0 - 0.9999$
RUN 20	512	$[42, 50]$	1.0
RUN 21	1024	$[10, 12]$	1.0
RUN 22	1024	$[1, 2]$	0

TABLE I: N: number of collocation points along each axis. k_f: range of forced wavenumbers. RUN 1-8: decimation of only negative helical modes with different probability in the range $\alpha \in [0 : 0.9999]$. RUN 9-13: same α-range of RUN 1-8 but with either positive or negative helical modes (with 50% probability) removed. RUN 14-19: similar to RUN 1-8 at higher resolutions. RUN 20-21: forced at small scales to observe the inverse energy cascade. RUN 22: same as RUN 1 at higher resolution.

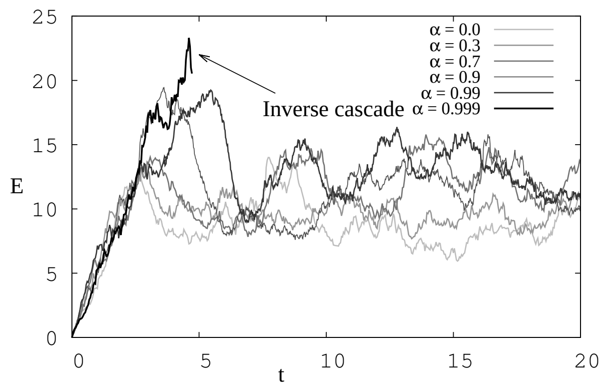

FIG. 1: Evolution of energy at varying α. Notice that all simulation reach a stationary state except for the case at $\alpha = 0.999$ where a constant increase of energy was observed signaling the existence of a robust stationary inverse energy transfer.

the decimated NSE in which a fraction α of the negative helical modes has been switched off [30]. We introduce the projector on positive/negative helical modes as

$$\mathcal{P}^{\pm}_{\boldsymbol{k}} \equiv \frac{\boldsymbol{h}^{\pm}_{\boldsymbol{k}} \otimes \overline{\boldsymbol{h}^{\pm}_{\boldsymbol{k}}}}{\overline{\boldsymbol{h}^{\pm}_{\boldsymbol{k}}} \cdot \boldsymbol{h}^{\pm}_{\boldsymbol{k}}}, \tag{3}$$

where $\overline{\bullet}$ denotes the complex conjugate. We define an operator D^{α} that projects each wavenumber with a probability $0 \leq \alpha \leq 1$:

$$\boldsymbol{u}^{\alpha}(\boldsymbol{x}) \equiv D^{\alpha}\boldsymbol{u}(\boldsymbol{x}) \equiv \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}}\, \mathcal{D}^{\alpha}_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}, \tag{4}$$

where $\mathcal{D}^{\alpha}_{\boldsymbol{k}} \equiv (1-\gamma^{\alpha}_{\boldsymbol{k}}) + \gamma^{\alpha}_{\boldsymbol{k}} \mathcal{P}^{+}_{\boldsymbol{k}}$ and $\gamma^{\alpha}_{\boldsymbol{k}} = 1$ with probability α or $\gamma^{\alpha}_{\boldsymbol{k}} = 0$ with probability $1 - \alpha$. The α-decimated Navier-Stokes equations (α-NSE) are

$$\partial_t \boldsymbol{u}^{\alpha} = D^{\alpha}[-\boldsymbol{u}^{\alpha} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{\alpha} - \boldsymbol{\nabla} p^{\alpha}] + \nu \Delta \boldsymbol{u}^{\alpha}, \tag{5}$$

where ν is the viscosity and p is the pressure. Notice that the nonlinear terms on the *rhs* of (5) are further projected by D^{α} in order to enforce the dynamics on the selected set of modes for all times. Despite of the fact that the α-NSE break the Lagrangian properties of the nonlinear terms [31], both energy and helicity:

$$E = \sum_{\boldsymbol{k}} (|u^{+}_{\boldsymbol{k}}|^2 + (1 - \gamma_{\boldsymbol{k}})|u^{-}_{\boldsymbol{k}}|^2), \tag{6}$$

$$H = \sum_{\boldsymbol{k}} k(|u^{+}_{\boldsymbol{k}}|^2 - (1 - \gamma_{\boldsymbol{k}})|u^{-}_{\boldsymbol{k}}|^2), \tag{7}$$

are still invariants in the inviscid limit, as one can readily derive from (5). We can then identify two extreme cases: when $\alpha = 0$ we recover the original NSE, when $\alpha = 1$ helicity becomes a coercitive quantity with a definite sign. It has been recently shown that in the latter case the dynamics of (5) develops a double cascade characterized by an inverse energy transfer with Kolmogorov spectrum $E(k) \sim k^{-5/3}$ for wavenumbers smaller than the forcing scale, $k \ll k_f$, and a direct helicity cascade with a $k^{-7/3}$ spectrum for $k \gg k_f$ [27, 28]. In this letter we address what happens in between, for $0 < \alpha < 1$. Does there exist a critical value, α_c, where the direction of the mean energy transfer suddenly reverse as observed in two-dimensional hydromagnetic systems at changing the forcing mechanisms [32]? Or the helicity plays a singular role? Are a few modes with opposite helical sign enough to transfer energy to small scales ($\alpha_c \to 1$) as suggested in [29] from considerations based on absolute equilibrium? What happens to small-scales intermittency in the forward cascade regime? Does it depend on the amount of negative/positive helical modes retained? Is the residual small-scales vorticity mainly helical? In order to answer all of these key questions we have performed a series of numerical simulations at changing α with a fully-dealiased, pseudo-spectral code at resolution up to 1024^3 on a triply periodic cubic domain of size $L = 2\pi$. The flow is sustained by a random Gaussian forcing with

$$\langle f_i(\boldsymbol{k}, t) f_j(\boldsymbol{q}, t')\rangle = F(k)\delta(\boldsymbol{k} - \boldsymbol{q})\delta(t - t')Q_{i,j}(\boldsymbol{k}),$$

where $Q_{ij}(\boldsymbol{k})$ is a projector assuring incompressibility and $F(k)$ has support only for $|k| \in [k_{\min} : k_{\max}]$ (see Table. I for details of the simulations). In all cases we have used a fully helical forcing with projection only on $\boldsymbol{h}^{+}_{\boldsymbol{k}}$ in order to ensure a maximal injection of helicity h independent of the degree of decimation α of negative helical modes.

We start by looking at the spectral properties of the system following [17]. We define the total spectra restricted to the positive/negative helical modes as $E^{+}(k) = \sum_{|\boldsymbol{k}|=k} |u^{+}_{\boldsymbol{k}}|^2; E^{-}(k) = \sum_{|\boldsymbol{k}|=k} (1 - \gamma_{\boldsymbol{k}})|u^{-}_{\boldsymbol{k}}|^2$ and the corresponding quantity for the helicity, $H^{\pm}(k) = k\, E^{\pm}(k)$. In the case when both energy and helicity are

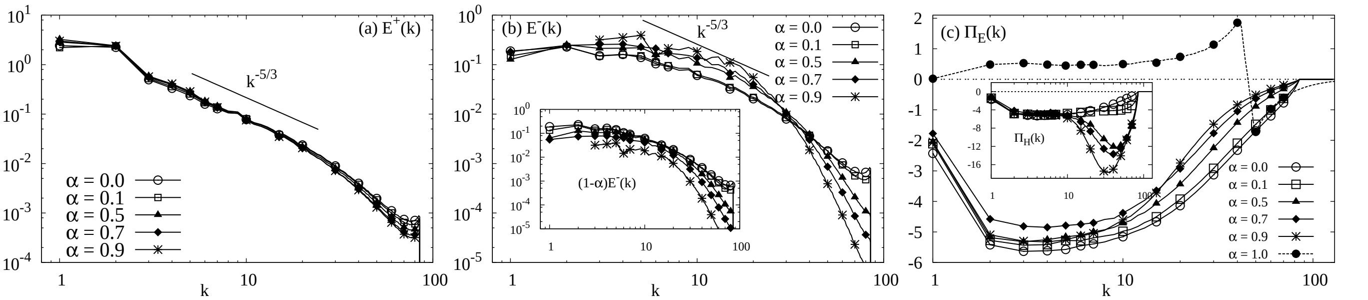

FIG. 2: (a) Log-log plot of $E^+(k) = \sum_{|\boldsymbol{k}|=k} |u^+_{\boldsymbol{k}}|^2$ vs k at changing α. (b) Log-log plot of $E^-(k) = \sum_{|\boldsymbol{k}|=k}(1-\gamma_{\boldsymbol{k}})|u^-_{\boldsymbol{k}}|^2$ vs k at changing α; Inset: rescaled $E^-(k)$ with factor $(1-\alpha)$. (c) Semi-log plot of flux of energy; Inset: flux of helicity, at changing α.

transferred forward with a rate ε and h respectively, we expect the usual Kolmogorov 1941 scaling (K41) for both energy and helicity spectra [12, 17]:

$$E(k) \sim C_E\ \varepsilon^{2/3}\ k^{-5/3}; \quad H(k) \sim C_H\ h\ \varepsilon^{-1/3} k^{-5/3},$$

which reflects in the scaling for each component as

$$E^{\pm}(k) = \varepsilon^{2/3} k^{-5/3} [1 \pm\ C\ h\ (\varepsilon k)^{-1}], \tag{8}$$

where $C = C_H/C_E$. In Fig. 1 we show the time evolution of the total energy E^α, given in (6), starting from a null configuration $\boldsymbol{u}_{\boldsymbol{k}} = 0$ at $t = 0$ at varying the degree of decimation from $\alpha = 0$, for the non-decimated NS case, to $\alpha \sim 1$. We notice first that the time needed to develop the initial release of energy becomes longer with increasing α and that the oscillations around the stationary regime, for long times, are also larger when $\alpha \sim 1$. The most striking phenomenon is that even for very high decimation of negative helical modes, $\alpha \sim 1$, the system is able to reach a stationary state transferring energy to the small-scales. In other words, it is enough to have a very few negative helical modes to develop a stable and stationary positive energy flux. This is quantified in Fig. 2 where we separately plot the spectra for the two helical components for various α. The spectrum for the positive helical modes (Fig. 2a) is almost unchanged and independent of α with a clearly developed $k^{-5/3}$ slope. Whereas the spectrum for the negative helical modes (Fig. 2b) tends to react back and become more and more energetic as α increases; this can be explained by looking at the behaviour of the energy flux. In Fig. 2c we show that the energy flux is constant and independent of α for all $\alpha < 1$, it reverts only for $\alpha \sim 1$. The surprising efficiency of the nonlinear transfer to find its way to small-scales suggests that helicity plays a singular role in turbulence: a tiny mixture of positive and negative helical modes, i.e., the existence of a few triads with mixed helicity signs, is enough to sustain energy transfer across all scales. This fact was suggested in [16] where the primary role of the triads with two high-wavenumber modes of opposite helicity was realized as the main contribution to the vortex stretching mechanisms. The constant energy flux must be mainly carried by triadic correlations with only one negative and two positive helical modes like

$$S(k|p,q) = \langle(\boldsymbol{k}\cdot u^-_{\boldsymbol{q}})(u^+_{\boldsymbol{k}}\cdot u^+_{\boldsymbol{p}})\rangle + \langle(\boldsymbol{k}\cdot u^+_{\boldsymbol{p}})(u^+_{\boldsymbol{k}}\cdot u^-_{\boldsymbol{q}})\rangle. \tag{9}$$

This is becasue such correlations are present with probability $\propto (1-\alpha)$ while other correlations, with two negative helical modes, are present with probability $\propto (1-\alpha)^2$ in the dynamics. Thus one can predict that

$$u^-_{\boldsymbol{k}} \to u^-_{\boldsymbol{k}}/(1-\alpha), \tag{10}$$

$$E^-(k) = \sum_{|\boldsymbol{k}|=k} (1-\gamma_{\boldsymbol{k}})|u^-_{\boldsymbol{k}}|^2 \to E^-(k)/(1-\alpha), \tag{11}$$

because each $u^-_{\boldsymbol{k}}$ in (9) must be renormalized by a factor $\propto 1/(1-\alpha)$ in order to keep the triadic correlation constant. As a result, negative helical modes retain more energy in order to maintain a constant energy flux. This prediction is shown to be well realized in the inset of Fig. 2b, where we show that rescaling $E^-(k)$ by a factor $(1-\alpha)$ leads to a good overlap except for $\alpha \sim 1$ where the fluctuations due to the onset of the inverse energy transfer becomes very large and the above argument possibly breaks down. Negative helical modes play a singular role. They act as ‘bridges’ for the energy transfer; they receive energy from the large-scale positive helical modes and release it to the small-scale positive helical modes; fewer they are more intense their amplitude must be to do it efficiently. Moreover, negative helical modes can transfer energy to other negative helical modes only if they form a triad; an event that has a probability $\propto (1-\alpha)^2$ to be present. When negative helical modes become too rare or absent, i.e., for $\alpha \sim 1$, this bridging is not possible anymore and the energy flows up-scale [27]. Helicity plays the role of a passive catalyst in the energy transfer. This can also be seen in the behavior of its flux (see inset of Fig. 2c) which is independent of α except at very high dissipative wavenumbers where the mismatch between energy of the positive and negative helical modes induce an increase of the helicity transfer [17, 18]. Proving the existence of a unique α_c for the inversion of the energy transfer could be extremely hard and it may not

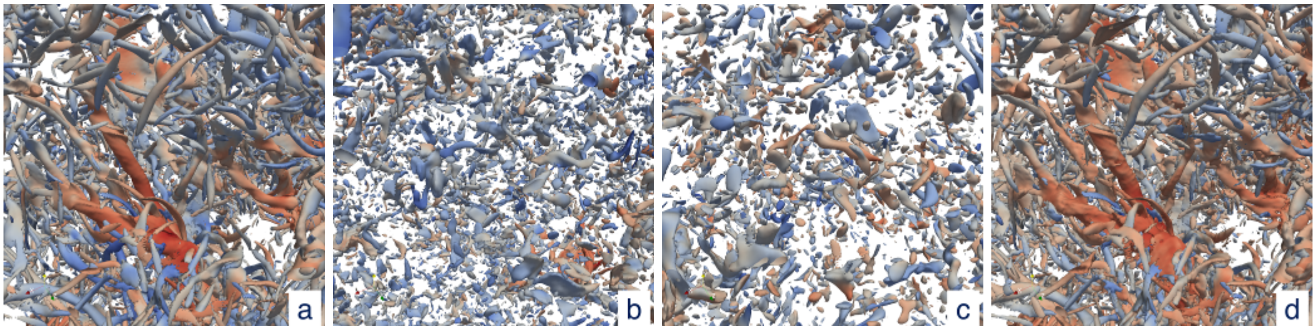

FIG. 3: (color online) iso-vorticity surfaces for: (a) $\alpha = 0$, (b) $\alpha = 0.5$, (c) $\alpha = 0.9$. Last plot (d) is obtained applying the projection with $\alpha = 0.5$ on the original NSE fields without any dynamical decimation. Color palette is proportional to the intensity of the helicity.

be crucial. The observed value is so close to unity that it might also be dependent on the realization of $\gamma_{\boldsymbol{k}}$ and/or on the Reynolds numbers. This issue is left for more detailed analysis in a future work.

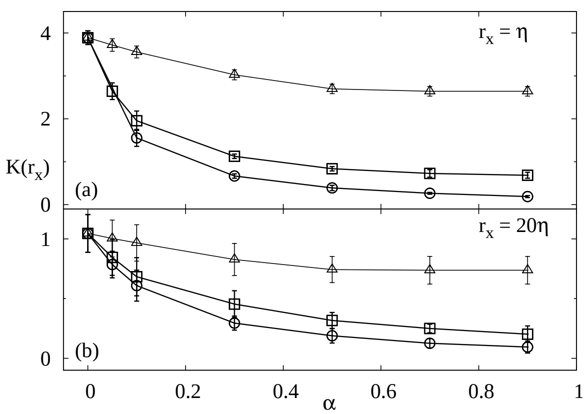

FIG. 4: Excess Kurtosis measured at the dissipative scale, $r_x = \eta$ (a) and in the inertial range, $r_x = 20\eta$ (b). (□): decimation of negative helical modes only; (◯): decimation of either positive or negative helical modes with 50% probability. (△): *aposteriori* decimation of negative helical modes from a velocity field of standard non-decimated NSE.

The second important problem addressed concerns with intermittency, the presence of strong non-Gaussian fluctuations at small scales, usually interpreted as a build up of instabilities in the vortex-stretching mechanisms. Here we want to understand how intermittency changes under the helical mode-reduction. A visual inspection of the vorticity field, in Fig. 3, shows a strong depletion of filament-like structures, starting from the standard 3D NSE (Fig. 3a), as a function of the degree of decimation of the negative helical modes (see Fig. 3b and Fig. 3c). In Fig. 4 we show the evolution of the excess Kurtosis,

$$K(r_x) = \frac{\langle (\delta_{r_x} u_y^\alpha)^4 \rangle}{\langle (\delta_{r_x} u_y^\alpha)^2 \rangle^2} - 3,$$

of the transverse velocity increments $\delta_{r_x} u_y^\alpha = u_y^\alpha(r_x) - u_y^\alpha(0)$ for two values of r_x and at changing α, where the selection of the $x - y$ components is arbitrary because of isotropy. We found that intermittency is very sensitive to α-decimation; it is enough to remove, from the dynamics, a small fraction of negative helical modes to strongly deplete the non-Gaussian character as measured by the fact that the excess Kurtosis is approaching 0. We show in Fig. 4 also the results of another numerical experiment, where we repeated the measurements in a set of simulations (RUN 9-13) with *random* decimation; this time either a positive *or* a negative helical mode is decimated with a global probability α. The reduction in the intensity of intermittency is comparable with the previous case; suggesting that it is mainly due to the decrease in the total number of dynamically active modes than due to their helical nature. This result is another manifestation of the passive role of helicity in the energy transfer mechanism. To further investigate the role of dynamic helical mode-reduction, we performed a projection *aposteriori*, applying the operator D^{alpha} to the velocity field obtained from a fully resolved non-decimated NSE ($\alpha = 0$). In this case, intermittency remains almost unchanged, independently of α, suggesting that only the dynamical mode-reduction is crucial to deplete the vortex stretching mechanism. For the original NSE positive and negative helical modes develop the same content of intermittency (see Fig. 3d for a visual confirmation of this fact). In conclusion, we have highlighted and quantified the singular role played by the helical Fourier modes in the energy flux reversal, showing that a forward transfer is always preferred as soon as a very small percentage of modes with opposite helicity are present. In contrast, the leading intermittent fluctuations are very fragile to any mode-reduction (helical or not helical) suggesting that the origin of real-space intermittency must rely on highly non-trivial and non-local correlations in Fourier space.

We acknowledge funding from the European Research Council under the European Union's Seventh Framework Programme, ERC Grant Agreement No 339032.

[1] D. Biskamp, *Magnetohydrodynamic Turbulence* (Cambridge University Press, Cambridge, UK, 2003).
[2] U. Frisch, *Turbulence: the legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, UK,1995).
[3] D. Lohse and K. Q. Xia, *Annu. Rev. Fluid Mech.* **42**, 335 (2010).
[4] R. H. Kraichnan, *Phys. Fluids* **10**, 1417 (1967).
[5] G. Boffetta and S. Musacchio, *Phys. Rev. E* **82**, 016307 (2010).
[6] J. Paret and P. Tabeling, *Phys. Fluids* **10**, 3126 (1998).
[7] H. J. H. Clercx, and G. J. F. van Heijst, *Appl. Mech. Rev.* **62**, 020802 (2009).
[8] P. Vorobieff, M. Rivera, and R. E. Ecke, *Phys. Fluids* **11**, 2167 (1999).
[9] M. Cencini, P. Muratore-Ginanneschi, and A. Vulpiani, *Phys. Rev. Lett.* **107**, 174502 (2011).
[10] H. K. Moffatt, *J. Fluid Mech.* **35**, 117 (1969).
[11] H. K. Moffatt and A. Tsinober, *Annu. Rev. Fluid Mech.* **24**, 281 (1992).
[12] A. Brissaud, U. Frisch, J. Leorat, M. Lesieur, and M. Mazure, *Phys. Fluids* **16**, 1366 (1973).
[13] C.E. Laing, R. L. Ricca, and D. W. L. Summers, *Scientific Reports* **5**, 9224 (2015).
[14] M. W. Scheeler, D. Kleckner, D. Proment, G.L. Kindlmann, and W.T.M. Irvine, *Proc. Natl. Acad. Sci.* **111**, 15350 (2014).
[15] R. H. Kraichnan, *J. Fluid Mech.* **47**, 525 (1971).
[16] F. Waleffe, *Phys Fluids A* **4**, 350 (1992).
[17] Q. Chen, S. Chen, and G. L. Eyink, *Phys. Fluids* **15**, 361 (2003);
[18] Q. Chen, S. Chen, G. L. Eyink, and D. D. Holm, *Phys. Rev. Lett.* **90**, 214503 (2003).
[19] P. D. Ditlevsen, *Phys. Fluids* **9**, 1482 (1997).
[20] R. Benzi, L. Biferale, R. M. Kerr, and E. Trovatore, *Phys. Rev. E* **53**, 3541 (1996).
[21] M. Lesieur and S. Ossia, *J. Turbulence* **1**, 7 (2000).
[22] P. D. Mininni, A. Alexakis, and A. Pouquet, *Phys. Fluids* **21**, 015108 (2009).
[23] E. Deusebio and E. Lindborg, *J. Fluid Mech.* **755**, 654 (2014).
[24] A. Celani, S. Musacchio, and D. Vincenzi, *Phys. Rev. Lett.* **104**, 184506 (2010).
[25] E. Herbert, F. Daviaud, B. Dubrulle, S. Nazarenko, and A. Naso, *Europhys. Lett.* **100**, 44003 (2012).
[26] A. Brandenburg, *Astrophysical J.* **550**, 824 (2001).
[27] L. Biferale, S. Musacchio, and F. Toschi, *Phys. Rev. Lett.* **108**, 164501 (2012).
[28] L. Biferale, S. Musacchio, and F. Toschi, *J. Fluid Mech.* **730**, 309 (2013).
[29] C. Herbert, *Phys. Rev. E* **89**, 013010 (2014).
[30] The choice of decimating negative mode is purely arbitrary nothing would change by decimating positive modes.
[31] H. K. Moffatt, *J. Fluid Mech.* **741**, R3 (2014).
[32] K Seshasayanan, S. J. Benavides, and A. Alexakis, *Phys. Rev. E* **90**, 051003 (2014).