

Multi-scale velocity correlations in turbulence and Burgerlence: fusion rules, Markov processes in scale, and multifractal predictions

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We compare different approaches towards an effective description of multi-scale velocity field correlations in turbulence. Predictions made by the operator product expansion, the so-called fusion rules, are placed in juxtaposition to an approach that interprets the turbulent energy cascade in terms of a Markov process of velocity increments in scale. We explicitly show that the fusion rules are a direct consequence of the Markov property provided that the structure functions exhibit scaling in the inertial range. Furthermore, the limit case of joint velocity gradient and velocity increment statistics is discussed and put into the context of the notion of dissipative anomaly. We generalize a prediction made by the multifractal (MF) approach derived in [Phys. Rev. Lett. 80, 3244 (1998)] to correlations among inertial range velocity increment and velocity gradients of any order. We show that for the case of squared velocity gradients such a relation can be derived from “first principles” in the case of Burgers equation. Our results are benchmarked by intensive direct numerical simulations of Burgers turbulence.

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I. INTRODUCTION

Three-dimensional turbulence is a paradigmatic out-of-equilibrium system with connections to fundamental questions in statistical mechanics [1, 2] and many other applied problems in different disciplines, e.g., mechanical engineering [3], atmospheric physics [4], geophysics [5], and astrophysics [6]. One of the most striking features of turbulence is that, already when stirred with a Gaussian, homogeneous and isotropic forcing, the flow develops highly non-trivial, non-Gaussian and multi-scale statistical properties in the limit of high Reynolds numbers. Here, the Reynolds number is the control parameter that defines the relative intensity of nonlinear vs. linear terms in the Navier-Stokes equation

$$\frac{\partial}{\partial t} \mathbf{v}(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t) \cdot \nabla \mathbf{v}(\mathbf{x}, t) = -\nabla p(\mathbf{x}, t) + \nu \nabla^2 \mathbf{v}(\mathbf{x}, t). \quad (1)$$

The existence of anomalous scaling properties goes under the name of intermittency, which is empirically found in all three-dimensional turbulent flows in nature and is still lacking a clear understanding and derivation from the underlying equations of fluid motion. Accordingly, this phenomenon of small-scale intermittency manifesting itself, e.g., in form of the non-self-similarity of the probability density function (PDF) of longitudinal velocity increments

$$\delta_r v = [\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})] \cdot \frac{\mathbf{r}}{r} \quad \text{for } r > 0, \quad (2)$$

is still one of the most compelling experimental, numerical and theoretical open problems of fully developed turbulence. Many studies of turbulence research have been

devoted to the experimental and theoretical examination of the scaling exponents $\zeta(n)$ of structure functions $\langle (\delta_r v)^n \rangle \sim r^{\zeta(n)}$ in the inertial range [1]. Here, Kolmogorov’s phenomenological description of the turbulent energy cascade, i.e., the transport process of energy from large to small scales, predicts $\zeta(n) = n/3$, which in turn implies a self-similar velocity increment PDF. The effects of intermittency lead to deviations from Kolmogorov’s theory and $\zeta(n)$ has been empirically found to be a non-linear function of n [1, 7–12].

The pivotal role of the turbulent energy cascade in turbulence theory immediately suggests the importance to extend the analysis based on single-scale observables (2) to multi-scale velocity increments, which should also lead to a better understanding of local and non-local correlations inside the inertial range and among inertial and viscous scales. Owing to the prohibitive analytical difficulties to attack the Navier-Stokes equation (1), the attention has been also often focused on other dynamical models of turbulence, in particular to the Burgers equations, a simplified one-dimensional and compressible version of the Navier-Stokes equation. Here, the only nonlinearity enters through the advective term

$$\frac{\partial}{\partial t} v(x, t) + v(x, t) \frac{\partial}{\partial x} v(x, t) = \nu \frac{\partial^2}{\partial x^2} v(x, t). \quad (3)$$

It is well known that the Burgers equation develops a quasi-shock for generic smooth initial conditions, a property that is also connected to anomalous scaling of the velocity increments [13]. In the following we will address both Navier-Stokes and Burgers equation using different statistical approaches to describe their multi-scale correlation properties, together with a series of quantitative validations using direct numerical simulations of Eq. (3).

In particular, we will compare the two seemingly different approaches of the *operator product expansion* [14–18] and the *Kramers-Moyal approach* [19–22]. It will be shown that both methods yield the *same* predictions for multi-scale velocity increment correlations, the so-called *fusion rules*. Subsequently, we will address the case where one of the increments matches the velocity gradient within the framework of the *multifractal approach* [1, 17, 23]. We will prove a particular expression of the multifractal (MF) approach from “first principles” in Burgers turbulence, i.e., by deriving an exact velocity increment hierarchy from the Burgers equation.

Historically, one of the first multi-scale analysis in turbulence was carried out in [14] where the *operator product expansion* from quantum field theory [24] was invoked. In this framework, one can derive the following relation for the two-increment (three-point) quantity

$$\langle (\delta_r v)^p (\delta_R v)^q \rangle \sim \frac{\langle (\delta_r v)^p \rangle}{\langle (\delta_R v)^p \rangle} \langle (\delta_R v)^{p+q} \rangle, \quad (4)$$

for $\eta < r \leq R \leq L$, where η is the dissipation scale and L the integral length scale. Moreover, we assume that one of the two extremes of the interval of length r and R coincide and that both increments are collinear. These relations, are known as *fusion rules*, and they have been analyzed both theoretically and numerically [15–18]. It should be noted that the fusion rules necessarily imply a reduction of the spatial complexity of the problem: the three-point quantity on the l.h.s. of Eq. (4) can be cast in terms of two-point quantities, the structure functions $\langle (\delta_r v)^n \rangle$. For three-dimensional isotropic and homogeneous turbulent flows, one can show [25] that the most general tensorial two-point velocity correlation function can be always decomposed in terms of longitudinal or transverse velocity structure functions. Here, for the sake of simplicity, we will always limit the discussion to the case when all distances are collinear with the velocity increments taken on the longitudinal direction as given by Eq. (2). Furthermore, this is the only possible case for one-dimensional Burgers turbulence (see below).

In the following, we will address the multi-scale correlation function (4) by using the MF approach [1, 17, 23] as well as the Kramers-Moyal (KM) approach [19–22] in order to describe the evolution of velocity increment PDFs across the inertial range. Within the MF approach we will also address multi-scale correlation functions when one of the velocity increment is calculated at fused-points, i.e., when the increment is smaller than the viscous dissipative cut-off. The latter case is important to discuss in the context of the so called *dissipative anomaly* [26] that emerges in a multi-point PDF hierarchy of Burgers turbulence (see also the discussion in Section IV A of this paper). Most of the theoretical arguments are general and can be applied both to the three-dimensional homogeneous and isotropic Navier-Stokes equation as well as to the one-dimensional Burgers equation. We will then present a series of detailed numerical benchmarks for the latter case only, where one

can achieve a separation of scales large enough to make precise quantitative statements. The paper is organized as follows: In Section II, we outline the usual derivation of the fusion rules (4) and discuss the dissipative cut-off within the framework of the MF approach. Henceforth, it will be shown in Section III that the fusion rules (4) can be derived from the KM expansion associated to a Markov process [27]. Section IV A contains a derivation of a multi-increment PDF hierarchy from Burgers equation which leads to a validation of the MF prediction from first principles. In the final part of this paper IV B, we will examine both fusion rules and the MF predictions in direct numerical simulations of Burgers turbulence.

II. FUSION-RULES AND THE MULTIFRACTAL APPROACH

The derivation of the fusion rules (4) starts from the assumption that the small-scale statistics of $\delta_r v$ is related to the large-scale configuration $\delta_R v$ via the multiplier $\lambda(r, R)$ according to

$$\delta_r v \sim \lambda(r, R) \delta_R v. \quad (5)$$

Furthermore, we assume that $\lambda(r, R) = \lambda(r/R)$, which is a consequence of a purely uncorrelated multiplicative process in addition to homogeneity along the energy cascade [17, 18] and yields

$$\begin{aligned} \langle (\delta_r v)^p (\delta_R v)^q \rangle &\sim \langle \lambda(r/R)^p (\delta_R v)^{p+q} \rangle \\ &\sim \langle \lambda(r/R)^p [\lambda(R/L) (\delta_L v)]^{p+q} \rangle, \end{aligned} \quad (6)$$

where we required that the large-scale increment is related to the integral scale increment by the same relation (5). Furthermore, $\delta_L v$ is assumed to be statistically independent from the multiplier $\lambda(r/L)$, which yields

$$\langle (\delta_r v)^p (\delta_R v)^q \rangle \sim \langle \lambda(r/R)^p \lambda(R/L)^{p+q} \rangle \langle (\delta_L v)^{p+q} \rangle, \quad (7)$$

but also implies that $\langle (\delta_r v)^p \rangle = \langle (\delta_L v)^p \rangle \langle \lambda(r/L)^p \rangle$. Hence, in the high-Reynolds number limit ($\text{Re} = \sqrt{\langle \mathbf{v}^2 \rangle} L / \nu \gg 1$, with the kinematic viscosity ν) where we expect scaling of the structure functions $\langle (\delta_r v)^p \rangle \sim (r/L)^{\zeta(p)}$, we can demand that $\langle \lambda(r/R)^p \rangle \sim (r/R)^{\zeta(p)}$. The last hypothesis that enters the derivation of the fusion rules (4) is that the multipliers obey an uncorrelated multiplicative process, which allows the splitting of the first expectation value on the r.h.s. of Eq. (7)

$$\begin{aligned} \langle (\delta_r v)^p (\delta_R v)^q \rangle &\sim \underbrace{\langle \lambda(r/R)^p \rangle}_{(r/R)^{\zeta(p)}} \underbrace{\langle \lambda(R/L)^{p+q} \rangle}_{\langle \lambda(R/L)^{p+q} (\delta_L v)^{p+q} \rangle} \langle (\delta_L v)^{p+q} \rangle \\ &\sim \frac{\langle (\delta_r v)^p \rangle}{\langle (\delta_R v)^p \rangle} \langle (\delta_R v)^{p+q} \rangle. \end{aligned} \quad (8)$$

In the following, we will also consider the case when the small-scale increment in Eq. (4) approaches the velocity gradient. On the basis of the MF approach one can deduce the existence of an *intermediate dissipation*

range [28], corresponding to a continuous range of dissipation lengths $\eta(h, \nu)$, where h denotes the continuous range of scaling exponents of the MF approach (see also [23]). In addition, the MF approach can be invoked in order to investigate the Reynolds number dependence of moments of velocity derivatives [29]. By the use of these multifractal calculations in combination with the intermediate dissipation range cut-off, one can derive expressions for joint velocity gradient-increment statistics [17, 18] such as

$$\left\langle \left(\frac{\partial v(x)}{\partial x} \right)^2 (\delta_R v(x))^q \right\rangle \sim \frac{R^{\zeta(q+3)-1}}{\nu}. \quad (9)$$

Here, we explicitly wrote the dependence of the increment $\delta_R v$ on x in order to indicate that the velocity gradient and the velocity increment are calculated with one point in common, x . Moreover, it must be stressed that this relation only holds if the scaling exponents fulfill Kolmogorov's 4/5-law, i.e., $\zeta(3) = 1$.

We now want to generalize the previous expression (9) to arbitrary orders of the velocity gradient. To this end, we define the quantity $D_{p,q}(\nu, R) = \langle (\partial v / \partial x)^p (\delta_R v)^q \rangle$ which can be written in terms of the dissipative scale, $\eta(\nu)$, as

$$D_{p,q}(\nu, R) = \left\langle \left(\frac{\delta_\eta v}{\eta} \right)^p (\delta_R v)^q \right\rangle. \quad (10)$$

The MF ansatz is based on the introduction of a set of scaling exponents h so that there exists a local scaling law

$$\delta_\eta v = (\eta/R)^h \delta_R v, \quad (11)$$

with probability $P_h(\eta, R) = (\eta/R)^{3-D(h)}$, where $D(h)$ is the fractal dimension of the set and where the velocity increment is Hölder continuous with exponent h (see also [1]). Furthermore, the dissipative scaling is defined by requiring an $O(1)$ local Reynolds number [28]

$$\text{Re}_\lambda = \frac{\eta \delta_\eta v}{\nu} \sim O(1). \quad (12)$$

As a result, we get a *fluctuating* η which depends on h and ν . Using (11) and (12) in (10), we obtain the first conditional expectation

$$\begin{aligned} & \left\langle \left(\frac{\delta_\eta v}{\eta} \right)^p (\delta_R v)^q \middle| \delta_R v \right\rangle \\ & \sim \int dh (\delta_R v)^{q+p} R^{-p} \left(\frac{\nu}{R \delta_R v} \right)^{\frac{p(h-1)+3-D(h)}{1+h}} \\ & \sim \frac{(\delta_R v)^{(q+p+\phi(p))} R^{\phi(p)-p}}{\nu^{\phi(p)}}, \end{aligned} \quad (13)$$

where we have used a saddle-point estimate in the limit of infinite Reynolds numbers, $\nu \rightarrow 0$, in order to get the exponent

$$\phi(p) = - \min_h \frac{p(h-1) + 3 - D(h)}{1+h}. \quad (14)$$

Finally we can estimate the unconditioned expectation value by considering again the MF ansatz to connect the velocity increment at scale R with the large scale velocity fluctuation v_L

$$\delta_R v = (R/L)^h \delta_L v \quad (15)$$

and integrating over all possible h

$$D_{p,q}(\nu, R) \sim \int dh R^{3-D(h)} \frac{(\delta_R v)^{(q+p+\phi(p))} R^{\phi(p)-p}}{\nu^{\phi(p)}}, \quad (16)$$

where we have taken $L = 1$ for simplicity. Plugging (15) in (16) and using again a saddle-point estimate in the limit $R \ll L = 1$, we get

$$D_{p,q}(\nu, R) \sim \text{Re}^{\phi(p)} R^{\zeta(p+q+\phi(p))} R^{\phi(p)-p}, \quad (17)$$

where the viscosity from Eq. (16) has been replaced by the dimensionless Reynolds number Re for which the relation $\text{Re} \sim O(1)/\nu$ holds. The exponents $\zeta(q)$ are the scaling exponents of the structure function of order q

$$\langle (\delta_R v)^q \rangle \sim \int dh (\delta_R v)^q R^{3-D(h)} \sim R^{\zeta(q)}, \quad (18)$$

with

$$\zeta(q) = \min_h (qh + 3 - D(h)). \quad (19)$$

It is important to remark that within the MF ansatz the scaling exponents of the velocity gradient, i.e., $\langle (\partial v / \partial x)^p \rangle \sim \text{Re}^{\phi(p)}$ and the structure function scaling exponent are connected via [1, 29]

$$\phi(p) = (\zeta(q) - q)/2 \quad \text{and} \quad p = (\zeta(q) + q)/2. \quad (20)$$

Using the above expression, it is easy to see that provided the third-order single-scale structure function satisfies the 4/5-law, $\zeta(3) = 1$, then for $p = 2$ the expression (17) possesses the remarkable property that it is inversely dependent on the viscosity ν , e.g., $\nu \langle (\partial v(x) / \partial x)^2 (\delta_R v(x))^q \rangle$ remains a finite quantity in the limit $\nu \rightarrow 0$, which is a sort of generalized *dissipative anomaly* [1].

In Section IV A we will prove Eq. (9) from "first principles" in Burgers turbulence and discuss the effects of pressure contribution that we have to face in the more general case of three-dimensional Navier-Stokes equation.

A different approach to the turbulent velocity gradient statistics was carried out recently [30–32]. Here, a series of order-dependent dissipative scales, η_{2n} are introduced starting from a balancing of inertial and diffusive terms of the equation for the $2n$ -th order longitudinal structure function

$$\eta_{2n} = L \text{Re}^{\frac{1}{\zeta(2n) - \zeta(2n+1) - 1}}. \quad (21)$$

Furthermore, the moments of the velocity gradient can be related to the structure functions via the local dissipation

Reynolds number (12) according to

$$\left\langle \left| \frac{\partial v}{\partial x} \right|^n \right\rangle \approx \left\langle \left| \frac{\delta_\eta v}{\eta} \right|^n \right\rangle = \frac{\langle (\delta_\eta v)^{2n} \rangle}{\nu} \sim \text{Re}^n \eta_{2n}^{\zeta(2n)}, \quad (22)$$

Eq. (21) implies Reynolds number scaling of the velocity gradients according to

$$\left\langle \left| \frac{\partial v}{\partial x} \right|^n \right\rangle = \text{Re}^{s_n}, \quad (23)$$

where

$$s_n = n + \frac{\zeta(2n)}{\zeta(2n) - \zeta(2n+1) - 1}. \quad (24)$$

The above prediction is different from the the MF result for $q = 0$ in Eq. (17), see also [33] for a quantitative comparison. Furthermore, it is not obvious how Eq. (23) should be generalized in order to predict the multi-scale dissipative-inertial correlation function (17). Finally, and more importantly, it is not consistent with the existence of a dissipative anomaly, i.e. with the constraint $s_2 = 1$, unless the relation $\zeta(5) - \zeta(4) = \zeta(4) - 1$ holds. Inserting $\zeta(3) = 1$, this relation suggests mono-scaling $\zeta(5) - \zeta(4) = \zeta(4) - \zeta(3)$, which is at odds with intermittency effects observed in three-dimensional turbulence (but compatible with the Burgers scaling, see below). As a result, the MF approach must yet be considered as the only description of multi-scale correlations in turbulence capable of reproducing the existence of dissipative anomaly. The latter depends only on the requirement that the exact 4/5-law is satisfied in the inertial range, i.e., $\zeta(3) = 1$.

III. MARKOV PROPERTY IN SCALE AND FUSION RULES

Another description of multi-increment statistics in turbulence was proposed in [19], using a *Markov process of velocity increments in scale* for the turbulent energy cascade. The latter approach starts from the definition of the n -increment PDF

$$f_n(v_n, r_n; v_{n-1}, r_{n-1}; \dots; v_1, r_1) = \prod_{i=1}^n \langle \delta(v_i - \delta_{r_i} v) \rangle. \quad (25)$$

where we restricted ourselves to longitudinal velocity increments (2) only (note that the inclusion of mixed longitudinal and transverse increment statistics necessarily complicates the entire procedure [34]). According to Bayes' theorem, we can define the conditional probabilities

$$p(v_3, r_3 | v_2, r_2; v_1, r_1) = \frac{f_3(v_3; r_3, v_2, r_2; v_1, r_1)}{f_2(v_2, r_2; v_1, r_1)}, \quad (26)$$

and

$$p(v_2, r_2 | v_1, r_1) = \frac{f_2(v_2, r_2; v_1, r_1)}{f_1(v_1, r_1)}. \quad (27)$$

Henceforth, the localness of interactions of the cascade process of the longitudinal velocity increments in scale is ensured by the Markov property in scale

$$p(v_3, r_3 | v_2, r_2; v_1, r_1) = p(v_3, r_3 | v_2, r_2), \quad (28)$$

where we assume that $\eta < r_3 \leq r_2 \leq r_1 < L$. The Markov property implies a considerable reduction of the spatial complexity of the velocity increment statistics, which can be deduced from the n -increment PDF (25): If one imposes the scale ordering $\eta < r_n \leq r_{n-1} \leq \dots \leq r_1 < L$, this $n + 1$ -point quantity factorizes due to the Markov property according to

$$f_n(v_n, r_n; v_{n-1}, r_{n-1}; \dots; v_1, r_1) = p(v_n, r_n | v_{n-1}, r_{n-1}) \times \dots \times p(v_2, r_2 | v_1, r_1) f_1(v_1, r_1), \quad (29)$$

Hence, the Markov property constitutes a *three-point-closure* of the multi-increment statistics [22, 35].

In the following, we want to examine the implications of (28) for the multi-scale moments (4). A central notion of a Markov process is that the transition PDF follows the same KM expansion as the one-increment PDF [27], namely

$$-\frac{\partial}{\partial r_2} f_1(v_2, r_2) = \hat{L}_{KM}(v_2, r_2) f_1(v_2, r_2), \quad (30)$$

$$-\frac{\partial}{\partial r_2} p(v_2, r_2 | v_1, r_1) = \hat{L}_{KM}(v_2, r_2) p(v_2, r_2 | v_1, r_1), \quad (31)$$

where the KM operator is defined as

$$\hat{L}_{KM}(v_2, r_2) = \sum_{k=1}^{\infty} (-1)^k \frac{\partial^k}{\partial v_2^k} D^{(k)}(v_2, r_2). \quad (32)$$

Furthermore, the minus sign in Eq. (31) indicates that the process occurs from large to small scales and the KM coefficients are defined as

$$D^{(k)}(v_2, r_2) = \frac{1}{k!} \lim_{r_3 \rightarrow r_2} \int dv_3 \frac{(v_3 - v_2)^k}{r_2 - r_3} p(v_3, r_3 | v_2, r_2). \quad (33)$$

The KM expansion (30) allows for an appealing formulation of intermittency via an evolution of the one-increment PDF (30) in scale. Moreover, scaling solutions for the structure functions, i.e., $\langle (\delta_r v)^n \rangle \sim r^{\zeta(n)}$ necessarily imply KM coefficients of the form [20, 21, 36]

$$D^{(k)}(v_2, r_2) = \frac{(-1)^k K_k v_2^k}{k! r_2}, \quad (34)$$

as can be seen by taking the moments $\int dv_2 v_2^n f(v_2, r) = \langle (\delta_r v)^n \rangle$ from Eq. (30) and setting $r_2 = r$

$$-\frac{\partial}{\partial r} \langle (\delta_r v)^n \rangle = \sum_{k=1}^n \binom{n}{k} K_k (-1)^k \frac{\langle \delta_r v^n \rangle}{r}, \quad (35)$$

Dividing by the structure function of order n yields

$$-\frac{\partial}{\partial r} \ln \langle (\delta_r v)^n \rangle = \frac{1}{r} \sum_{k=1}^n \binom{n}{k} K_k (-1)^k. \quad (36)$$

model	scaling exponent $\zeta(n)$	reduced KM coefficients K_n
K41	$n/3$	$K_1 = 1/3$, no higher orders
K62 ^a	$n/3 - \mu n(n-3)/18$	$K_1 = (3 + \mu)/9$, $K_2 = \mu/9$, no higher orders
Burgers-ramps	n	$K_1 = 1$, no higher orders
Burgers-shocks	1	$K_n = 1$
β -model ^b	$\frac{D_F-2}{3}n + (3 - D_F)$	$K_1 = \frac{D_F-2}{3} + (3 - D_F)$, $K_n = 3 - D_F$ for $n > 1$
She-Leveque ^c	$\frac{n}{9} + 2 \left(1 - \left(\frac{2}{3}\right)^{n/3}\right)$	$K_n = \frac{1}{9} \left(n {}_1F_0(1-n; ; 1) + 18 \left(1 - \sqrt[3]{\frac{2}{3}}\right)^n \right)$
Yakhot	$\frac{(1+3\beta)n}{3(1+\beta n)}$	$K_n = \frac{\Gamma[n+1]}{\Gamma[n+1+\frac{1}{\beta}]} \left(\Gamma[1 + \frac{1}{\beta}] + \frac{1}{3\beta^2} \Gamma[\frac{1}{\beta}] \right)$

TABLE I. Phenomenological models of turbulence (we refer the reader to [20, 21] for further discussions) with scaling exponents $\zeta(n)$ and the corresponding reduced KM coefficients from Eq. (34): Kolmogorov’s mean field theory from 1941 (K41), Kolmogorov-Oboukhov theory from 1962 (K62), Burgers phenomenology, β -model, She-Leveque phenomenology, and Yakhot’s model. Note that the K41 as well as the Burgers-ramps do not exhibit intermittency corrections. The K62 phenomenology is the only intermittency model that can be reproduced with just two KM coefficients.

^a $\mu \approx 0.227$, ^b $D_F \approx 2.83$, ^c here, ${}_pF_q(a; b; z)$ is the generalized hypergeometric function, ^d $\beta = 0.05$

Integrating this equation from r to L yields

$$\langle (\delta_r v)^n \rangle = \langle (\delta_L v)^n \rangle \left(\frac{r}{L} \right)^{-\sum_{k=1}^n \binom{n}{k} K_k (-1)^k}. \quad (37)$$

Accordingly, the reduced KM coefficients K_k are related to the scaling exponents $\zeta(n)$ according to

$$\zeta(n) = - \sum_{k=1}^n \binom{n}{k} K_k (-1)^k. \quad (38)$$

All currently known phenomenological models of turbulence are reproduced by a suitable choice of the reduced KM coefficients listed in Tab. I. Another important implication of this KM description of structure function scaling follows directly from the moment solution (37): In order to obtain non-vanishing odd order moments (such as Kolmogorov’s 4/5-law $\langle \delta_r v^3 \rangle = -\frac{4}{5} \langle \varepsilon \rangle r$) at a scale r one must have *non-vanishing odd order moments* at large scales L . In other words, the symmetric form of the KM expansion dictated by the coefficients (34) is not able to generate skewness during the cascade process, it can only transport an initial large-scale skewness in the PDF ”down in the cascade”.

In the original works [19, 37, 38] the KM expansion (30) was truncated after the second coefficient which reduces the expansion to an ordinary Fokker-Planck equation (consistent with K62 scaling, see Tab. I). This truncation is motivated by Pawula’s theorem [27], which states that if an even order KM coefficient $n > 2$ is zero then all other coefficients $n > 2$ are zero as well. In this particular case, it can be shown [39, 40] that multi-scale correlations obey fusion rules (4). However, the restriction to a Fokker-Planck equation based on the Pawula theorem has proven to be a questionable approximation [20–22] and higher-order coefficients were found to be small but non-vanishing (see also Table I). We will show below that

the fusion rules are valid even considering the entire KM expansion. To this end, we cast the solution of Eq. (31) in form of a Dyson series [27] replacing $r_2 = r$ and $r_1 = R$

$$\begin{aligned} & p(v_2, r | v_1, R) \\ &= \delta(v_2 - v_1) + \int_r^R dr_1 \hat{L}_{KM}(v_2, r_1) \delta(v_2 - v_1) \\ &+ \int_r^R dr_1 \int_r^{r_1} dr_2 \hat{L}_{KM}(v_2, r_1) \hat{L}_{KM}(v_2, r_2) \delta(v_2 - v_1) \\ &+ \dots \\ &= \delta(v_2 - v_1) + \int_r^R dr_1 \frac{\hat{L}(v_2)}{r_1} \delta(v_2 - v_1) \\ &+ \int_r^R dr_1 \int_r^{r_1} dr_2 \frac{\hat{L}(v_2)^2}{r_1 r_2} \delta(v_2 - v_1) + \dots \\ &= \delta(v_2 - v_1) + \ln \frac{R}{r} \hat{L}(v_2) \delta(v_2 - v_1) \\ &+ \frac{1}{2!} \left(\ln \frac{R}{r} \right)^2 \hat{L}(v_2)^2 \delta(v_2 - v_1) + \dots \\ &= \exp \left[\ln \frac{R}{r} \hat{L}(v_2) \right] \delta(v_2 - v_1), \end{aligned} \quad (39)$$

where the scale-independent differential operator $\hat{L}(v_2)$ is defined according to

$$\hat{L}(v_2) = \sum_{k=1}^{\infty} \frac{K_k}{k!} \frac{\partial^k}{\partial v_2^k} v_2^k. \quad (40)$$

Note that the ”scale ordering problem” in the first line of the Dyson series (39) can be omitted due to the separable form of the KM coefficients (34).

We are now in the position to introduce the three-point moments (4). Due to the ordering $r \leq R$, we can take the moments of the two-increment PDF $f_2(v_2, r; v_1, R) =$

$p(v_2, r|v_1, R)f_1(v_1, R) = \langle \delta(v_2 - \delta_r v)\delta(v_1 - \delta_R v) \rangle$ and obtain

$$\begin{aligned} & \langle (\delta_r v)^p (\delta_R v)^q \rangle \\ &= \int dv_2 v_2^p \int dv_1 v_1^q p(v_2, r|v_1, R) f_1(v_1, R). \end{aligned} \quad (41)$$

Inserting the Dyson series (31) for the transition PDF $p(v_2, r|v_1, R)$ yields

$$\begin{aligned} \langle (\delta_r v)^p (\delta_R v)^q \rangle &= \langle (\delta_R v)^{p+q} \rangle + \ln \frac{R}{r} \sum_{k=1}^{\infty} \frac{K_k}{k!} \int dv_1 v_1^q \int dv_2 v_2^p \frac{\partial^k}{\partial v_2^k} v_2^k \delta(v_2 - v_1) f_1(v_1, R) \\ &+ \frac{1}{2!} \left(\ln \frac{R}{r} \right)^2 \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{K_k K_l}{k! l!} \int dv_1 v_1^q \int dv_2 v_2^p \frac{\partial^k}{\partial v_2^k} v_2^k \frac{\partial^l}{\partial v_2^l} v_2^l \delta(v_2 - v_1) f_1(v_1, R) + \dots \end{aligned} \quad (42)$$

Partial integrations with respect to v_2 in the second and

third term yield

$$\begin{aligned} \langle (\delta_r v)^p (\delta_R v)^q \rangle &= \langle (\delta_R v)^{p+q} \rangle + \ln \frac{R}{r} \sum_{k=1}^p \frac{(-1)^k K_k p!}{k! (p-k)!} \int dv_1 v_1^q \int dv_2 v_2^p \delta(v_2 - v_1) f_1(v_1, R) \\ &+ \frac{1}{2!} \left(\ln \frac{R}{r} \right)^2 \sum_{k=1}^p \frac{(-1)^k K_k p!}{k! (p-k)!} \sum_{l=1}^p \frac{(-1)^l K_l p!}{l! (p-l)!} \int dv_1 v_1^q \int dv_2 v_2^p \delta(v_2 - v_1) f_1(v_1, R) + \dots \\ &= \left[1 + \ln \frac{R}{r} \sum_{k=1}^p (-1)^k K_k \binom{p}{k} + \frac{1}{2!} \left(\ln \frac{R}{r} \right)^2 \sum_{k=1}^p (-1)^k K_k \binom{p}{k} \sum_{l=1}^p (-1)^l K_l \binom{p}{l} + \dots \right] \langle (\delta_R v)^{p+q} \rangle \\ &= \left[1 - \ln \frac{R}{r} \zeta(p) + \frac{1}{2!} \left(\ln \frac{R}{r} \right)^2 \zeta(p)^2 + \dots \right] \langle (\delta_R v)^{p+q} \rangle = \exp \left[-\zeta(p) \ln \frac{R}{r} \right] \langle (\delta_R v)^{p+q} \rangle \\ &= \exp \left[\zeta(p) \ln \frac{r}{R} \right] \langle (\delta_R v)^{p+q} \rangle = \frac{r^{\zeta(p)}}{R^{\zeta(p)}} \langle (\delta_R v)^{p+q} \rangle = \frac{\langle (\delta_r v)^p \rangle}{\langle (\delta_R v)^p \rangle} \langle (\delta_R v)^{p+q} \rangle. \end{aligned} \quad (43)$$

Here, we made use of relation (38) and inserted $\langle (\delta_r v)^p \rangle \sim r^{\zeta(p)}$ in the last step. In other words, the operator product expansion can be conceived as a Markov process of velocity increments in scale, a direct consequence of the multiplicative process (5) and its uncorrelated multipliers. Empirical evidences suggest that the multiplicative uncorrelated fusion-rules prediction (43) breaks down in the limit of $r \rightarrow R$. In terms of the Markov property (28), such a violation can be explained by the existence of non-trivial correlations in the energy transfer for not too separated scales.

In conclusion, the application of the fusion rules (4) necessarily entails two aspects:

- i.) the validity of the Markov property of velocity increments in scale (28), which implies that the KM expansion for the transition PDF (31) conforms with the KM expansion for the one-increment PDF (30).
- ii.) the specific form of the KM coefficients (34) which was chosen in a way to ensure the existence of scaling solutions $\langle \delta_r v^n \rangle \sim r^{\zeta(n)}$.

For the sake of completeness, we want to end this section with a generalization of fusion rules (4) to n -increment statistics ($n+1$ -point statistics in terms of ordinary moments). The procedure follows along the same lines than the derivation of the fusion rules from the KM expansions of the Markov process (43) and is explained in Appendix A. We obtain

$$\begin{aligned} & \langle (\delta_{r_n} v)^{p_n} \dots (\delta_{r_2} v)^{p_2} (\delta_{r_1} v)^{p_1} \rangle \\ &= \int dv_n \dots dv_2 dv_1 v_n^{p_n} \dots v_2^{p_2} v_1^{p_1} f_n(v_n, r_n; \dots v_1, r_1) \\ &= \prod_{i=1}^{n-1} \frac{\langle (\delta_{r_{i+1}} v)^{\sum_{k=1}^i p_{n+1-k}} \rangle}{\langle (\delta_{r_i} v)^{\sum_{k=1}^i p_{n+1-k}} \rangle} \langle (\delta_{r_1} v)^{\sum_{k=1}^n p_{n+1-k}} \rangle, \end{aligned} \quad (44)$$

where f_n is the n -increment PDF (25). These generalized fusion rules imply a reduction of a $n+1$ -point statistical quantity to a two-point quantity.

IV. APPLICATION TO BURGERS TURBULENCE

In contrast to the dissipation anomaly that arises in the MF description (Section II), the dissipation anomaly that arises in the multi-scale description of Burgers turbulence bears a clear physical meaning: Due to absence of nonlocal pressure contributions, singular structures consist of localized shocks whose widths are determined by the viscosity ν . E.g., consider the single shock solution of Eq. (47)

$$v(x, t) = 1 - \tanh\left(\frac{x - x_c - t}{2\nu}\right), \quad (45)$$

where the width of the shock is inversely proportional to ν . It can be readily seen that the averaged local energy dissipation rate $\langle \varepsilon \rangle$, where

$$\varepsilon(x) = 2\nu \left(\frac{\partial v(x)}{\partial x}\right)^2 \quad (46)$$

is independent of the viscosity ν . In the following, we will further discuss multi-scale properties of the Burg-

ers equations, including inertial-viscous cases as the ones described by the correlations (17).

A. Dissipation anomaly in a multi-increment PDF hierarchy in Burgers turbulence

We consider the Burgers equation

$$\frac{\partial}{\partial t} v(x, t) + v(x, t) \frac{\partial}{\partial x} v(x, t) = \nu \frac{\partial^2}{\partial x^2} v(x, t) + F(x, t), \quad (47)$$

with a white-noise in time Gaussian forcing $F(x, t)$ defined by the second order moment

$$\langle F(x, t) F(x', t') \rangle = \chi(x - x') \delta(t - t'), \quad (48)$$

where $\chi(x - x')$ is the spatial correlation function, assumed to be concentrated around a characteristic scale $|x - x'| \sim l_f$. The evolution equation for the velocity increment $\delta_r v(x, t)$ is

$$\begin{aligned} & \frac{\partial \delta_r v(x, t)}{\partial t} + v(x, t) \frac{\partial \delta_r v(x, t)}{\partial x} + \delta_r v(x, t) \frac{\partial \delta_r v(x, t)}{\partial r} \\ &= \nu \frac{\partial^2 \delta_r v(x, t)}{\partial x^2} + F(x + r, t) - F(x, t). \end{aligned} \quad (49)$$

The temporal evolution of the one-increment PDF (2) is derived in Appendix B according to

$$\begin{aligned} & \frac{\partial}{\partial t} f_1(v_1, r_1, t) + v_1 \frac{\partial}{\partial r_1} f_1(v_1, r_1, t) + 2 \int_{-\infty}^{v_1} dv'_1 \frac{\partial}{\partial r_1} f_1(v'_1, r_1, t) \\ &= -\nu \frac{\partial}{\partial v_1} \int dr_2 [\delta(r_2 - r_1) - \delta(r_2)] \frac{\partial^2}{\partial r_2^2} \int dv_2 v_2 f_2(v_2, r_2; v_1, r_1, t) + [\chi(0) - \chi(r_1)] \frac{\partial^2}{\partial v_1^2} f_1(v_1, r_1, t). \end{aligned} \quad (50)$$

Due to the viscous coupling to the two-increment PDF, we have a hierarchy formally similar to the BBGKY statistical physics case [26, 41].

It is useful to reformulate the dissipative terms in order to introduce the local energy dissipation rate (46). First, we assume the stationarity of the velocity increment statistics, i.e., $\frac{\partial}{\partial t} f_1(v_1, r_1, t) = 0$. Second, as shown in Appendix C, the unclosed viscous term in Eq. (50) can be rewritten in terms of the joint velocity gradient and velocity increment statistics as

$$\begin{aligned} v_1 \frac{\partial}{\partial r_1} f_1(v_1, r_1) &= -2 \int_{-\infty}^{v_1} dv'_1 \frac{\partial}{\partial r_1} f_1(v'_1, r_1) \\ &- \frac{\partial^2}{\partial v_1^2} \left[\left\langle \frac{\varepsilon(x)}{2} [\delta(v_1 - \delta_{r_1} v(x)) + \delta(v_1 + \delta_{-r_1} v(x))] \right\rangle \right. \\ &\left. + [\chi(0) - \chi(r_1)] f_1(v_1, r_1) \right] + 2\nu \frac{\partial^2}{\partial r_1^2} f_1(v_1, r_1). \end{aligned} \quad (51)$$

From the above expression, the existence of the dissipa-

tive anomaly becomes more apparent than in Eq. (50) due to the non-vanishing local energy dissipation rate in the limit $\nu \rightarrow 0$. Taking the moments of Eq. (51) and dropping the index of r_1 yields

$$\begin{aligned} & \left(1 - \frac{2}{n}\right) \frac{\partial}{\partial r} \langle (\delta_r v(x))^n \rangle \\ &= 2\nu \frac{\partial^2}{\partial r^2} \langle (\delta_r v(x))^{n-1} \rangle \\ &- \frac{(n-1)(n-2)}{2} \langle \varepsilon(x) [(\delta_r v(x))^{n-3} + (-\delta_{-r} v(x))^{n-3}] \rangle \\ &+ (n-1)(n-2) [\chi(0) - \chi(r)] \langle (\delta_r v(x))^{n-3} \rangle. \end{aligned} \quad (52)$$

For $n = 3$, we recover the equivalent of Kolmogorov's 4/5-law for Burgers turbulence

$$\begin{aligned} \frac{1}{3} \frac{\partial}{\partial r} \langle (\delta_r v(x))^3 \rangle &= -2\langle \varepsilon \rangle + 2\nu \frac{\partial^2}{\partial r^2} \langle (\delta_r v(x))^2 \rangle \\ &+ 2[\chi(0) - \chi(r)], \end{aligned} \quad (53)$$

which reduces to $\langle (\delta_r v)^3 \rangle = -6\langle \varepsilon \rangle r$ in the inertial range.

In the general case, i.e., for $n \neq 3$, we start by discarding the forcing contribution in the inertial range $\eta \ll r \ll L$ in assuming that $\chi(r)$ decreases sufficiently fast for increasing r . Moreover, in the limit of high Reynolds numbers, i.e., $\nu \rightarrow 0$, the smooth subleading viscous term $2\nu \frac{\partial^2}{\partial x^2} \langle (\delta_r v)^{n-1} \rangle$ can be neglected. Hence, in the inertial range where $\langle (\delta_r v^{n-3}) \rangle$ should admit scaling, we obtain

$$\langle \varepsilon(x) (\delta_r v(x))^n \rangle \sim |r|^{\zeta(n+3)-1}, \quad (54)$$

which agrees with the first result (9) of the MF approach. Hence, the prediction made by the MF approach (9) be-

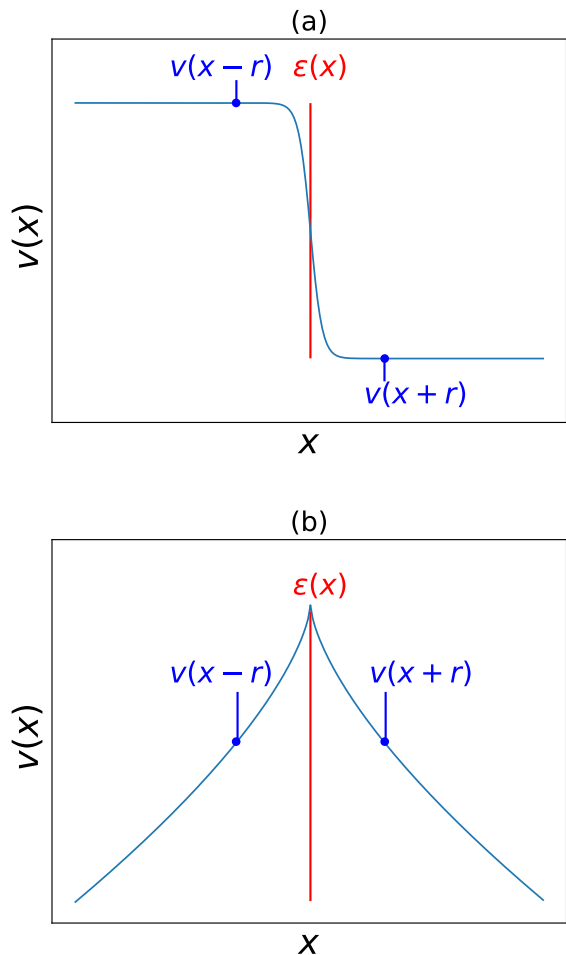


FIG. 1. (a) Schematic depiction of a shock in Burgers turbulence. The local energy dissipation rate is peaked at the center of the shock $\varepsilon(x)$. Depending on the strength of the shock, the velocity field at $v(x-r)$ and at $v(x+r)$ possesses the symmetry $v(x-r) = -v(x+r)$, which leads to Burgers scaling (55). (b) In the case of cusp-like structures, $\varepsilon(x)$ is still peaked in the center of the cusp. The symmetry of the cusps, however, leads to the vanishing of the dissipation anomaly in Eq. (52).

comes *exact* for the case of Burgers turbulence, something not known before to the best of our knowledge. It

must be stressed that (52) does not further specify the scaling exponent $\zeta(n)$. It is well known that in order to go beyond it, we need some heuristic arguments about the dissipative term based on the geometrical structures of the flow. In high-Reynolds number Burgers turbulence, we are faced with shock-like structures similar to the one in Fig. (1)(a). In this case, the local energy dissipation rate is peaked at the center of the shock and the velocities $v(x+r)$ and $v(x-r)$ are arranged anti-symmetrically around $v(x) = 0$. In the limit of small viscosities and for small r , $v(x \pm r)$ possesses a negligible dependence on r and we obtain

$$\begin{aligned} & \langle \varepsilon(x) [(\underbrace{\delta_r v(x)}_{=v(x+r)})^{n-3} + (\underbrace{-\delta_{-r} v(x)}_{=-v(x-r)})^{n-3}] \rangle \\ & \sim \langle v^2 \rangle^{(n-3)/2} \langle \varepsilon \rangle \sim r^{\zeta(n)-1} \rightarrow \zeta(n) = 1. \end{aligned} \quad (55)$$

This is exactly the celebrated Burgers-shock scaling from Table (I).

The influence of smooth velocity field structures can be seen as follows: Consider Eq. (52) for small r , in which case, we can neglect the nonlinear and forcing contributions.

$$\begin{aligned} & 2\nu \frac{\partial^2}{\partial r^2} \langle (\delta_r v(x))^{n-1} \rangle \\ & \approx \frac{(n-1)(n-2)}{2} \langle \varepsilon(x) [(\delta_r v(x))^{n-3} + (-\delta_{-r} v(x))^{n-3}] \rangle \\ & \approx 2\nu(n-1)(n-2) \left\langle \left(\frac{\partial v(x)}{\partial x} \right)^{n-1} r^{n-3} \right\rangle, \end{aligned} \quad (56)$$

where we performed a Taylor expansion $\delta_r v(x) = v(x+r) - v(x) \approx \frac{\partial v(x)}{\partial x} r$ inside the ensemble average on the r.h.s. and replaced the local energy dissipation rate $\varepsilon(x)$ by its definition (46). Integrating Eq. (56) and inserting back the definition of the local energy dissipation rate (46) yields

$$\langle (\delta_r v(x))^n \rangle = 2^{-n/2-1/2} \frac{\langle \varepsilon^{n/2} \rangle}{\nu^{n/2-1/2}} r^n. \quad (57)$$

Obviously, this result bears the signature of smooth ramp-like velocity field contributions $u(x)$ in between shocks and is the leading term for $n < 1$. Hence, by including the heuristic result (55), we obtain the well-known Burgers scaling

$$\langle |\delta_r v|^n \rangle \sim \begin{cases} r^n & \text{for } n < 1, \\ r & \text{for } n \geq 1. \end{cases} \quad (58)$$

In order to understand the importance of the exact shape of the singularity, it is instructive to consider the case of the Burgers equation with an additional nonlocality [21, 42].

$$\frac{\partial}{\partial t} v(x, t) + w(x, t) \frac{\partial}{\partial x} v(x, t) = \nu \frac{\partial^2}{\partial x^2} v(x, t) + F(x, t), \quad (59)$$

where the convective velocity field is given by

$$w(x, t) = \alpha v(x, t) + (1 - \alpha) \text{p.v.} \int dx' \frac{v(x', t)}{x - x'}. \quad (60)$$

Here, $\alpha = 1$ corresponds to the case of Burgers turbulence, whereas $\alpha = 0$ corresponds to the purely non-local case that exhibits self-similar behavior [42]. In the latter case, the velocity field is dominated by cusp-like structures similar to the one depicted in Fig. 1 (b). Consequently, the velocity field possesses the symmetry $v(x-r) = v(x+r)$ leading to the vanishing of the dissipative term $\langle \varepsilon(x) [(\delta_r v(x))^{n-3} + (-\delta_{-r} v(x))^{n-3}] \rangle$ for even n . Furthermore, the nonlinear terms in the PDF hierarchy are changed due to the presence of the nonlocality in the generalized Burgers equation (59) and are necessarily unclosed [22]. Accordingly, the nonlinear terms in the purely nonlocal case are balanced by the forcing terms. Depending on the properties of the forcing correlation function this scaling can be associated with the results of the renormalization group (see [43] for further references) and necessarily implies non-intermittent scaling.

Another important case of Eq. (52) is when the local dissipation rate and the velocity increment are statistically independent

$$\begin{aligned} & \langle \varepsilon(x) [(\delta_r v(x))^{n-3} + (-\delta_{-r} v(x))^{n-3}] \rangle \\ &= \underbrace{\langle [(\delta_r v(x))^{n-3} + (-\delta_{-r} v(x))^{n-3}] \rangle}_{\sim r_1^{\zeta(n-3)}} \langle \varepsilon(x) \rangle \\ &\sim r^{\zeta(n)-1} \rightarrow \zeta(n) - 1 = \zeta(n-3) \rightarrow \zeta(n) = n/3. \end{aligned} \quad (61)$$

which necessarily implies K41 scaling.

The case of Burgers scaling (55) must be considered as the opposite case: the energy dissipation rate is fully correlated with the velocity increment, leading to strong intermittency. Furthermore, it has been shown that the intermediate case $0 < \alpha < 1$ in Eq. (59) bears many resemblances with the original Navier-Stokes equation [21, 42]. Accordingly, the pressure must have a regularizing effect on the velocity field structures that enter the dissipation anomaly.

In the following section, we will evaluate both the fusion rules from Section III as well as the multifractal prediction from direct numerical simulations of Burgers turbulence.

B. Direct numerical simulations of Burgers turbulence

In order to validate the theoretical considerations of the previous sections, we performed direct numerical simulations (DNS) of the stochastically driven Burgers equation (47). The numerical setup consists of a second order Adams-Bashforth explicit solver paired with an Euler-Maruyama step to account for the large-scale Gaussian

run	#1	#2	#3
u_{rms}	1.16	1.16	1.15
ν	$3.6 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$	$6.8 \cdot 10^{-3}$
Re	1800	550	90
Re_λ	100	56	23
$\langle \varepsilon \rangle$	1	1	1
dt	$1.53 \cdot 10^{-5}$	$1.53 \cdot 10^{-5}$	$1.53 \cdot 10^{-5}$
dx	$3.83 \cdot 10^{-4}$	$3.83 \cdot 10^{-4}$	$3.83 \cdot 10^{-4}$
η	$2.61 \cdot 10^{-3}$	$6.31 \cdot 10^{-3}$	$2.37 \cdot 10^{-2}$
λ	0.031	0.056	0.134
L	1.564	1.555	1.526
T in T_L	760	762	772
N	2^{14}	2^{14}	2^{14}
k_F	5	5	5

TABLE II. Characteristic parameters of the numerical simulations: root mean square velocity $v_{rms} = \sqrt{\langle v^2 \rangle}$, viscosity ν , averaged rate of local energy dissipation $\langle \varepsilon \rangle = 2\nu \langle (\frac{\partial v}{\partial x})^2 \rangle$, grid spacing dx, timestep dt, dissipation length $\eta = (\frac{\nu^3}{\langle \varepsilon \rangle})^{1/4}$, Taylor length $\lambda = v_{rms} \sqrt{\frac{2\nu}{\langle \varepsilon \rangle}}$, Taylor-Reynolds number $\text{Re}_\lambda = \frac{v_{rms} \lambda}{\nu}$, integral length scale $L = \frac{E_{kin}^{3/2}}{\langle \varepsilon \rangle}$, kinetic energy $E_{kin} = \frac{1}{2} v_{rms}^2$, large-eddy turn-over time $T_L = \frac{L}{v_{rms}}$, number of grid points N , and k_F is the maximum forced wavenumber of the power law forcing. The physical domain size is 2π .

random forcing. We also consider the variable transformation, $\hat{v}'_k(t) = \exp(-\nu k^2 dt) \hat{v}_k(t)$, which implies the exact integration of the viscous term. It relaxes the restriction on the time step by the diffusive term and significantly improves the convergence for large wave numbers. The spatial correlation function of the forcing (48) follows a power law $\propto k^{-2}$ in Fourier space and has a cut-off at $k_F = 5$. Table II contains a list of the characteristic parameters in use for the simulations presented in Figs. 3-5. The resolution was fixed such that $\eta/dx \approx 6$ at the highest Reynolds number. To improve the statistics we averaged over two hundred independent runs.

1. Evaluation of inertial-inertial fusion rules from DNS of Burgers turbulence

First, we investigate the validity of the fusion rules (4) for the Burgers equation. To this end, we consider the quantity

$$F_{p,q}(r, R) = \langle (\delta_r v)^p (\delta_R v)^q \rangle. \quad (62)$$

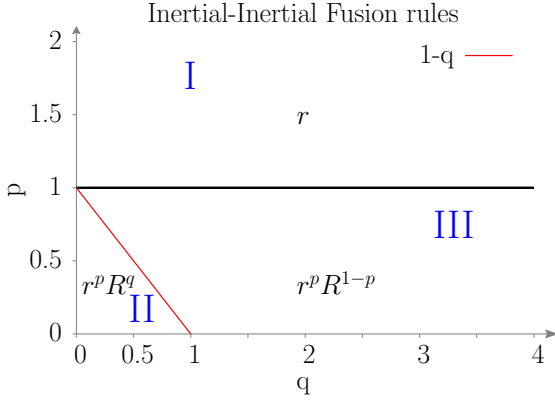


FIG. 2. Parametric space of the exponents of Eq. (62) and the corresponding fusion rules prediction. I-III correspond to the three regions that emerge from the bifractal Burgers scaling (58).

The application of the fusion rules (4) in conjunction with Burgers scaling (58) yields three different possible scaling properties, depending on the order of the moments p, q . If both increments are dominated by the shock we have case (I), if both are dominated by the smooth ramps we have the case (II), while if the small scale is smooth and the large scale is dominated by the shock we have case (III). The scaling prediction in the plane (p, q) is summarized in Fig. (2) and in Eq. (63).

$$F_{p,q}(r, R) \sim \begin{cases} r & \text{for } p > 1, \\ r^p R^q & \text{for } p < 1 \text{ \& } p + q \leq 1, \\ r^p R^{1-p} & \text{for } p < 1 \text{ \& } p + q \geq 1. \end{cases} \quad (63)$$

In the following, we fix the large scale R to $\pi/2$ and vary the small scale r . R is fixed so that $R/\eta \approx 600$ for $\text{Re} = 1800$, $R/\eta \approx 250$ for $\text{Re} = 550$, and $R/\eta \approx 70$ for $\text{Re} = 90$. We have also tested the opposite scenario by fixing the small scale r to $\alpha\eta$ with $\alpha \in [2, 10]$, which yielded similar results that will therefore not be shown here. Fig. 3 depicts $F_{p,q}(r, R)$ for three values in the three regions of Fig. 2: $p = 2, q = 4$ (top, region I), $p = q = 0.4$ (center, region II), and $p = 0.6, q = 2$ (bottom, region III). As one can see, all three cases agree fairly well with the theoretical predictions (black lines with corresponding predicted scaling). This becomes even more apparent from the insets in Fig. 3, which shows $F_{p,q}(r, R)$ compensated by the corresponding prediction. We observe constant (r -independent) regions over a few decades of r/R . However, as r approaches larger values and tends towards R , the compensated function becomes r -dependent, which indicates a breakdown of the fusion rules for small scale separations. As discussed in Section III, the breakdown of the fusion rules for small scale separations can also be interpreted in terms of the violation of the Markov property (28). In the following section, we will consider the special case of $F(r, R)$ for $r \rightarrow \eta$ to check the viscous-inertial scaling.

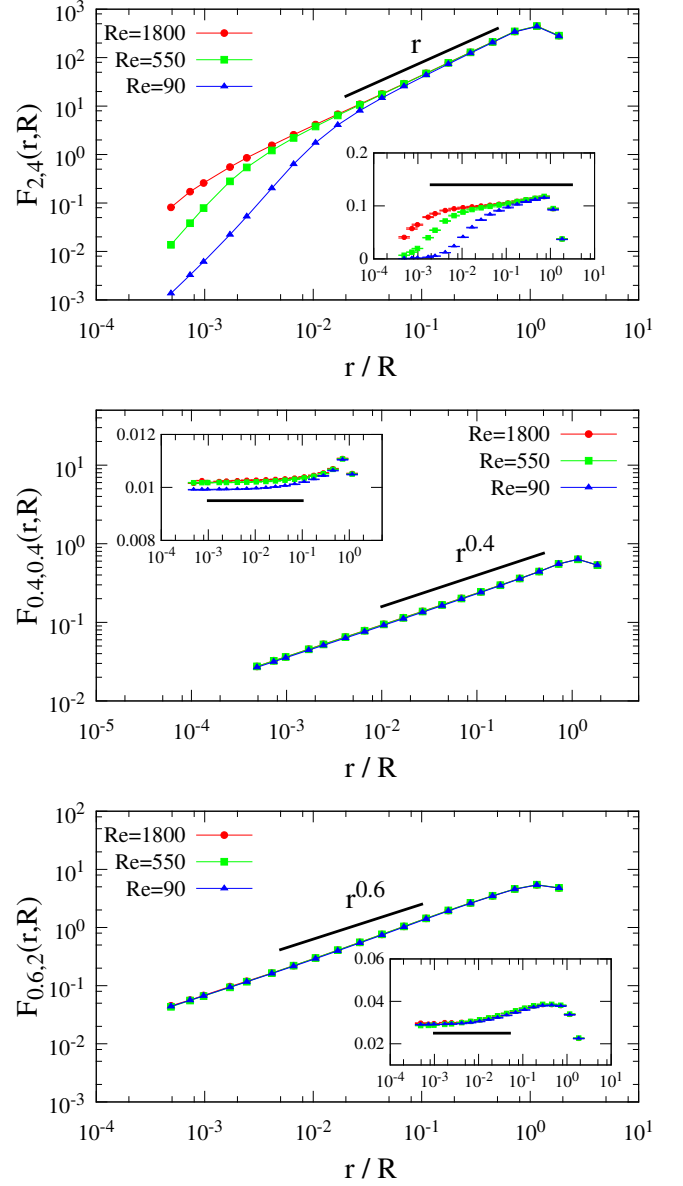


FIG. 3. Examination of the inertial-inertial fusion rules Eq. (63) for runs #1-#3 via the quantity $F_{p,q}(r, R)$ defined in Eq. (62) as a function of r/R for different values of p and q and for different Reynolds numbers. The black line corresponds to the fusion rules prediction in Burgers turbulence summarized in the three regions I-III of Fig. 2. The inset depicts $F_{p,q}(r, R)$ divided by this prediction. *Top*: $p = 2$ and $q = 4$ (region I). *Center*: $p = q = 0.4$ (region II). *Bottom*: $p = 0.6$ and $q = 2$ (region III).

2. Evaluation of the viscous-inertial fusion rules prediction from DNS of Burgers turbulence

In the following, we consider the viscous-inertial multi-scale correlation function given

$$D_{p,q}(\text{Re}, R) = \langle (\partial_x v)^p (\delta_R v)^q \rangle \quad (64)$$

We specialize to the Burgers case for which the MF prediction is given in Fig. 4 and in Eq. (65) by inspecting Eq. (17).

$$D_{p,q}(\text{Re}, R) \sim \begin{cases} \text{Re}^{p-1} & \text{for } p > 1, \\ R^q & \text{for } p < 1 \text{ \& } p + q \leq 1, \\ R^{1-p} & \text{for } p < 1 \text{ \& } p + q \geq 1. \end{cases} \quad (65)$$

In the following, we will also refer to these relations as the MF prediction for the *viscous-inertial fusion rules*.

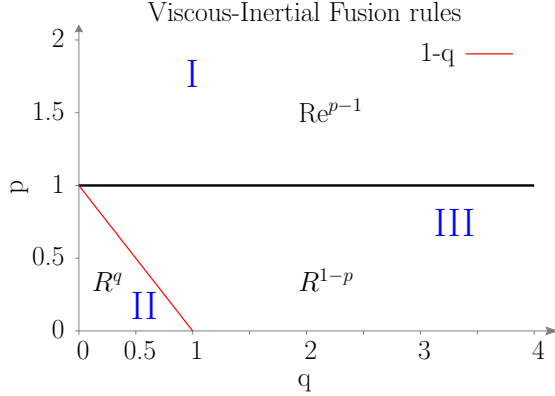


FIG. 4. Parametric space of the exponents of Eq. (64) and the MF prediction for the viscous-inertial fusion rules. I-III correspond to the three regions that emerge from the bifractal Burgers scaling (58).

The top panel in Fig. 5 depicts $\nu D_{2,4}(\text{Re}, R)$ as a function of R for different Reynolds numbers (see also Table II). As it can be seen, $\nu D_{2,4}(\text{Re}, R)$ is independent of the scale R in the inertial range, which is in accordance with the MF prediction in region I of Fig. 4. The collapse of the data comes in agreement with the existence of a generalized dissipative anomaly as discussed in Section II and Eq. (9). The flat region increases as $\nu \rightarrow 0$. The middle and the bottom panels in Fig. 5 depict $D_{0.4,0.4}(\text{Re}, R)$ and $D_{0.6,2}(\text{Re}, R)$ respectively, with combinations of exponents p, q corresponding to regions II and III of Fig. 4, accordingly. The black line with scaling R^m , $m \in \mathbb{R}$, is the viscous-inertial fusion rules prediction following Eq. (65) for the chosen exponents p, q . The inset plot depicts the data divided by R^m . All three cases show a good agreement of the data with the viscous-inertial fusion rules prediction which improves as we increase Re.

3. Evaluation of the velocity gradient statistics from DNS of Burgers turbulence

Finally, we want to consider the special case where Eq. (64) reduces to the ordinary moments of the velocity gradient, i.e., $D_{p,0}(\text{Re})$. As it can be seen from Fig. 4, the MF prediction for Burgers reduces to the Reynolds

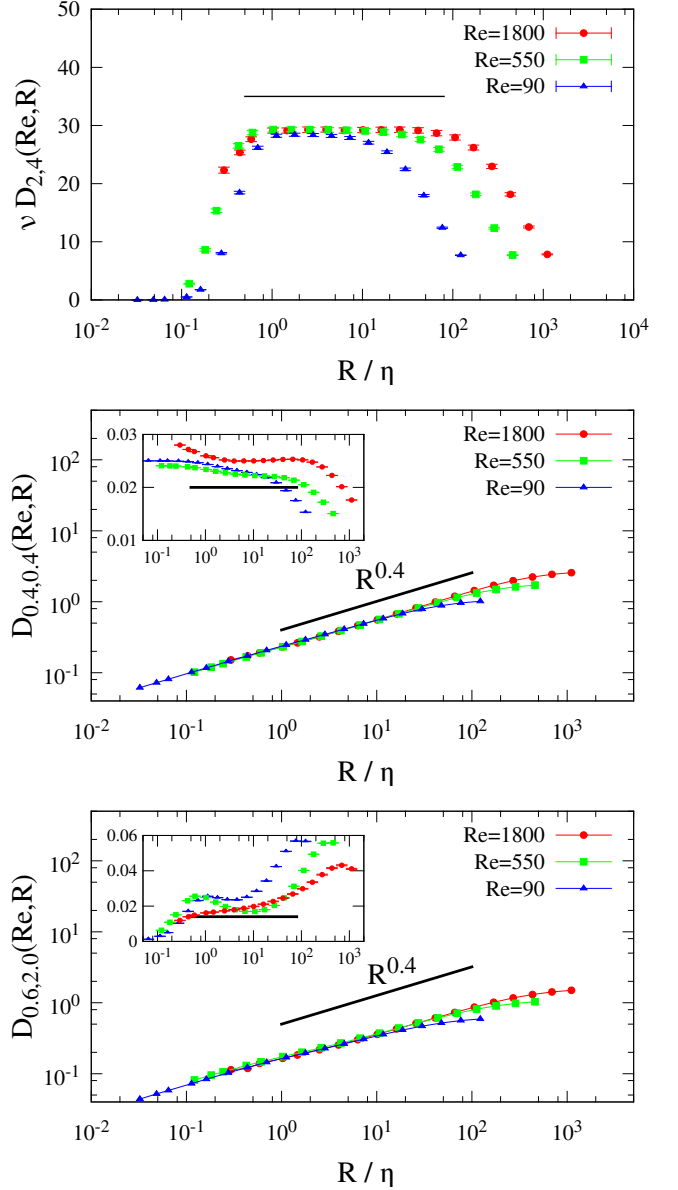


FIG. 5. Examination of the viscous-inertial fusion rules prediction Eq. (65) via the quantity $D_{p,q}(\text{Re}, R)$ defined in Eq. (64) as a function of R/η for different values of p and q and for different Reynolds numbers. The inset plots depict the results divided by the corresponding viscous-inertial fusion rules prediction of regions I-III of Fig. 4. *Top*: $p = 2$ and $q = 4$ (region I). The quantity $\nu D_{2,4}(\text{Re}, R)$ is shown here. It is independent of R , and the collapse of the data within errorbars supports the suggestion of a generalized dissipative anomaly discussed in Sec. II and Eq. (9). *Center*: $p = q = 0.4$ (region II). *Bottom*: $p = 0.6$ and $q = 2$ (region III).

number scaling

$$D_{p,0} \sim \text{Re}^{p-1} \quad \text{for } p > 1. \quad (66)$$

Moreover, for the particular case of Burgers turbulence with $\zeta(n) = 1$, both the MF prediction (17) and the result from [30–32] in Eq. (23) yields relation (66), which

was already discussed in Section II. It is convenient to introduce the quantity

$$M_p = \frac{\left\langle \left(\frac{\partial v}{\partial x} \right)^p \right\rangle}{\left\langle \left(\frac{\partial v}{\partial x} \right)^2 \right\rangle^{p/2}} \quad (67)$$

for even p . Recent numerical investigations of hydrodynamic turbulence [44] suggest that the moments (67) exhibit a transition from Gaussian to anomalous behavior if one increases the Reynolds number. Hence, we expect M_p to behave according to

$$M_p \sim \begin{cases} (p-1)!! & \text{for } \text{Re} \sim O(1), \\ \text{Re}^{p/2-1} \sim \text{Re}_\lambda^{p-2} & \text{for } \text{Re} \gg O(1) \end{cases} \quad (68)$$

for even p . Here, we made use of the fact that the Taylor-Reynolds number $\text{Re}_\lambda = u_{rms}\lambda/\nu$ is related to Re according to $\text{Re} \sim \text{Re}_\lambda^2$ in the high-Reynolds number regime.

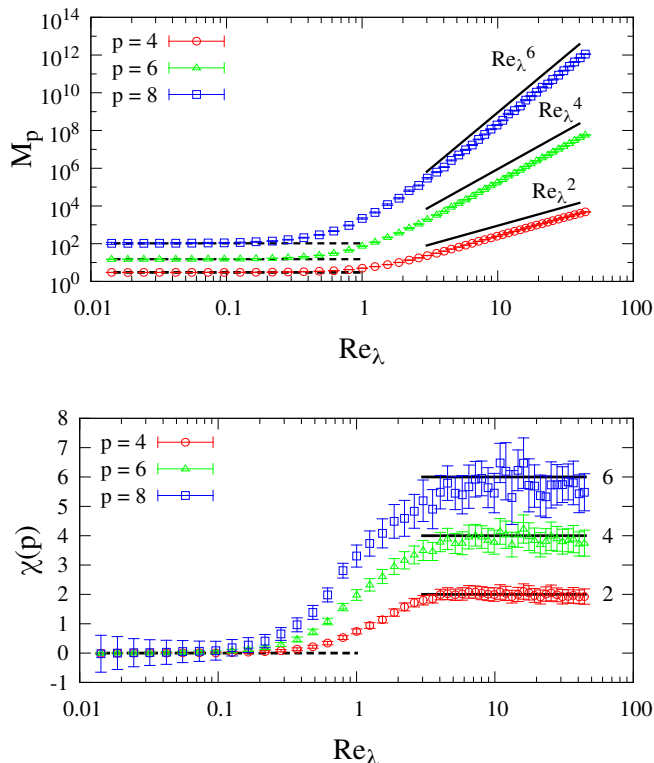


FIG. 6. Top: Moments M_p from Eq. (67) as a function of the Taylor-Reynolds number in Burgers turbulence. The low-Reynolds number regime exhibits Gaussian statistics (dashed black lines), whereas the high-Reynolds number regime agrees well with the multifractal/Yakhot prediction (filled black line) for Burgers turbulence (68). The lines correspond to flat regions of the logarithmic derivative of the moments $\chi(n)$, see panel below. Bottom: Logarithmic derivative of the moments (69). The straight black lines correspond to the theoretical predictions (68), $\chi(4) = 2$, $\chi(6) = 4$, and $\chi(8) = 6$

Fig. 6 is in quantitative agreement with Eqs. (68). The top panel depicts the moments (67) as a function of the Taylor-Reynolds number. For small Re_λ , the moments exhibit Gaussian statistics similar to the case of hydrodynamic turbulence [44], whereas the anomalous behavior for larger Re_λ is much more pronounced in comparison to the latter case. Obviously, this result can be attributed to the strong intermittency behavior in Burgers turbulence. Nevertheless, in the high-Reynolds number regime we can confirm prediction (68) to a great extent. The fits (black lines) in the top of Fig. 6 correspond to flat regions in the logarithmic derivative of the moments (67)

$$\chi(p) = \frac{d \log M_p}{d \log \text{Re}_\lambda}, \quad (69)$$

which is displayed in the bottom panel of Fig. 6. The flat regions are indicated as flat lines which correspond to the theoretical predictions (68), $\chi(4) = 2$, $\chi(6) = 4$, and $\chi(8) = 6$. Hence, we can conclude that the MF prediction also applies to the single gradient statistics in Burgers turbulence.

V. CONCLUSIONS

We have presented an overview of prevalent concepts that allow for multi-scale descriptions of turbulent flows. A main result of this paper is that the operator product expansion/fusion rules approach [14–16, 26] that emanated from quantum field theory is a direct consequence of the Markov property of velocity increments in scale devised in [19], provided that the structure functions exhibit scaling in the inertial range. This means an amalgamation of two fields that co-existed for nearly 20 years. By contrast, our results might also lead to a novel *stochastic* interpretation of the operator product expansion in quantum field theory [24]. At difference from other closure methods, e.g. the quasi-normal approximation [45], renormalization methods [46, 47], renormalization group methods [48], eddy-damped quasi-normal approximation [43, 49–51], both the Markov approach and the operator product expansion are *non-perturbative*, i.e., are not based on properties of Gaussian-distributed velocity field fluctuations. The latter property makes both approaches suitable candidates for a closure of the multi-increment PDF hierarchy [22].

Regarding the breakdown of the fusion rules in the limit of small scale separations, it is tempting to investigate the influence of non-Markovian cascade processes. Here, a generalization of the KM expansion for the transition PDF (31) to arbitrary stochastic processes as emphasized in [52], might yield a generalization of the fusion rules to arbitrary cascade processes. A dissipative cut-off of the structure functions [53] can also be achieved by a dissipative KM expansion and will be presented elsewhere. In addition, the Markov property could be considered as a first step in an approximation of multi-increment statistics. The natural next step would be an

extension incorporating one additional level of "memory in scale" [22], e.g. assuming

$$p(v_4, r_4 | v_3, v_3; r_2, r_2; v_1, r_1) \approx p(v_4, r_4 | v_3, v_3; r_2, r_2) \quad (70)$$

and thus allowing to capture correlations between the inertial and viscous-inertial range.

Furthermore, we have shown that a specific prediction of the MF model for joint velocity gradient and velocity increment statistics (9) can be obtained from the basic fluid dynamical equations under the neglect of pressure contributions, i.e., from the Burgers equation. It must be stressed that this result can be derived *without* any further assumptions apart from the scaling of structure functions in the inertial range. However, at this point, we could not validate the generalization of the MF result to arbitrary powers of the velocity gradient given by Eq. (13). In order to derive such a generalization, one has to operate at the next level of the multi-increment hierarchy (50). Here, a possible closure is the Markov property (28) which leads to a self-consistent equation for the two-increment PDF [22].

The numerical part of this work was devoted to the verification of fusion rules and the prediction of the MF prediction in DNS of Burgers turbulence. Both fusion rules and MF prediction could be established to a certain extend. The limitation of the fusion rules arises for vanishing scale separations and could be understood from the violation of the Markov property (28). A further examination of this regime will be the task of future research.

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Appendix A: Generalization of fusion rules to n -increment statistics

We consider the moments of the n -increment PDF

$$\begin{aligned} & \langle (\delta_{r_n} v)^{p_n} \dots (\delta_{r_2} v)^{p_2} (\delta_{r_1} v)^{p_1} \rangle \quad (A1) \\ &= \int dv_n \dots dv_2 dv_1 v_n^{p_n} \dots v_2^{p_2} v_1^{p_1} f_n(v_n, r_n; \dots; v_1, r_1), \end{aligned}$$

where the p_i 's denote arbitrary exponents and were we impose the scale ordering $\eta \leq r_n \leq r_{n-1} \leq \dots \leq r_2 \leq r_1 \leq L$. First, we rewrite the n -increment PDF according to Bayes' theorem

$$\begin{aligned} & f_n(v_n, r_n; \dots; v_1, r_1) \\ &= p(v_n, r_n | v_{n-1}, r_{n-1}; \dots; v_1, r_1) \\ & \quad \times f_{n-1}(v_{n-1}, r_{n-1}; \dots; v_1, r_1). \quad (A2) \end{aligned}$$

The general form of the Markov property in scale implies that

$$p(v_n, r_n | v_{n-1}, r_{n-1}; \dots; v_1, r_1) = p(v_n, r_n | v_{n-1}, r_{n-1}). \quad (A3)$$

Hence, Eq. (A1) simplifies to

$$\begin{aligned} & \int dv_n \dots dv_2 dv_1 v_n^{p_n} \dots v_2^{p_2} v_1^{p_1} p(v_n, r_n | v_{n-1}, r_{n-1}) \\ & \quad \times f_{n-1}(v_{n-1}, r_{n-1}; \dots; v_1, r_1) \quad (A4) \end{aligned}$$

Under the assumption of the scaling of structure functions in combination with the Markov property, we can express the conditional probability $p(v_n, r_n | v_{n-1}, r_{n-1})$ in terms of a Dyson series (39)

$$\begin{aligned} p(v_n, r_n | v_{n-1}, r_{n-1}) &= \delta(v_n - v_{n-1}) + \ln \frac{r_{n-1}}{r_n} \sum_{k=1}^{\infty} \frac{K_k}{k!} \frac{\partial^k}{\partial v_n^k} v_n^k \delta(v_n - v_{n-1}) \\ & \quad + \frac{1}{2!} \left(\ln \frac{r_{n-1}}{r_n} \right)^2 \sum_{k=1}^{\infty} \frac{K_k}{k!} \frac{\partial^k}{\partial v_n^k} v_n^k \sum_{l=1}^{\infty} \frac{K_l}{l!} \frac{\partial^l}{\partial v_n^l} v_n^l \delta(v_n - v_{n-1}) + \dots \quad (A5) \end{aligned}$$

Inserting (A5) into (A4) and performing the partial in-

tegrations with respect to v_n similar to Eq. (43) yields

$$\left[1 + \sum_{k=1}^{p_n} (-1)^k K_k \binom{p_n}{k} \ln \frac{r_{n-1}}{r_n} + \frac{1}{2!} \sum_{k=1}^{p_n} (-1)^k K_k \binom{p_n}{k} \sum_{l=1}^{p_n} (-1)^l K_l \binom{p_n}{l} \left(\ln \frac{r_{n-1}}{r_n} \right)^2 + \dots \right] \quad (\text{A6})$$

$$\int dv_{n-1} \dots dv_2 dv_1 v_{n-1}^{p_n+p_{n-1}} \dots v_2^{p_2} v_1^{p_1} f_{n-1}(v_{n-1}, r_{n-1}; \dots; v_1, r_1).$$

Here, the square bracket can be written as an exponential function according to

$$\left[\dots \right] = \exp \left[\ln \frac{r_{n-1}}{r_n} \sum_{k=1}^{p_n} (-1)^k K_k \binom{p_n}{k} \right]. \quad (\text{A7})$$

The sum in the exponential function can be identified as the scaling exponent $\zeta(p_n) = -\sum_{k=1}^{p_n} (-1)^k K_k \binom{p_n}{k}$ and we obtain

$$\begin{aligned} & \langle (\delta_{r_n} v)^{p_n} (\delta_{r_{n-1}} v)^{p_{n-1}} \dots (\delta_{r_2} v)^{p_2} (\delta_{r_1} v)^{p_1} \rangle \\ &= \exp \left[\ln \frac{r_n}{r_{n-1}} \zeta(p_n) \right] \\ & \times \langle (\delta_{r_{n-1}} v)^{p_n+p_{n-1}} \dots (\delta_{r_2} v)^{p_2} (\delta_{r_1} v)^{p_1} \rangle, \quad (\text{A8}) \end{aligned}$$

Furthermore, the scaling of the structure functions implies that $\langle (\delta_{r_n} v)^{p_n} \rangle \sim r_n^{\zeta(p_n)}$, which yields

$$\begin{aligned} & \langle (\delta_{r_n} v)^{p_n} (\delta_{r_{n-1}} v)^{p_{n-1}} \dots (\delta_{r_2} v)^{p_2} (\delta_{r_1} v)^{p_1} \rangle \quad (\text{A9}) \\ &= \frac{\langle (\delta_{r_n} v)^{p_n} \rangle}{\langle (\delta_{r_{n-1}} v)^{p_n} \rangle} \langle (\delta_{r_{n-1}} v)^{p_n+p_{n-1}} \dots (\delta_{r_2} v)^{p_2} (\delta_{r_1} v)^{p_1} \rangle. \end{aligned}$$

Successive application of this relation yields

$$\begin{aligned} & \langle (\delta_{r_n} v)^{p_n} (\delta_{r_{n-1}} v)^{p_{n-1}} \dots (\delta_{r_2} v)^{p_2} (\delta_{r_1} v)^{p_1} \rangle \\ &= \frac{\langle (\delta_{r_n} v)^{p_n} \rangle}{\langle (\delta_{r_{n-1}} v)^{p_n} \rangle} \times \frac{\langle (\delta_{r_{n-1}} v)^{p_n+p_{n-1}} \rangle}{\langle (\delta_{r_{n-2}} v)^{p_n+p_{n-1}} \rangle} \times \dots \\ & \times \frac{\langle (\delta_{r_2} v)^{p_n+\dots+p_2} \rangle}{\langle (\delta_{r_1} v)^{p_n+\dots+p_2} \rangle} \times \langle (\delta_{r_1} v)^{p_n+\dots+p_1} \rangle, \quad (\text{A10}) \end{aligned}$$

or in a more compact notation

$$\begin{aligned} & \prod_{i=1}^n \langle (\delta_{r_i} v)^{p_i} \rangle \quad (\text{A11}) \\ &= \prod_{i=1}^{n-1} \frac{\langle (\delta_{r_{i+1}} v)^{\sum_{k=1}^i p_{n+1-k}} \rangle}{\langle (\delta_{r_i} v)^{\sum_{k=1}^i p_{n+1-k}} \rangle} \langle (\delta_{r_1} v)^{\sum_{k=1}^n p_{n+1-k}} \rangle, \end{aligned}$$

which is the counterpart to Eq. (45).

Appendix B: Derivation of multi-increment hierarchy in Burgers turbulence

In order to derive the evolution equation (50) we take the temporal derivative of the one-increment PDF

$$\begin{aligned} & \frac{\partial}{\partial t} f_1(v_1, r_1, t) = \frac{\partial}{\partial t} \langle \delta(v_1 - \delta_{r_1} v(x, t)) \rangle = -\frac{\partial}{\partial v_1} \left\langle \delta(v_1 - \delta_{r_1} v(x, t)) \frac{\partial}{\partial t} \delta_{r_1} v(x, t) \right\rangle \quad (\text{B1}) \\ &= \frac{\partial}{\partial v_1} \left\langle \delta(v_1 - \delta_{r_1} v(x, t)) \left[v(x, t) \frac{\partial}{\partial x} \delta_{r_1} v(x, t) + \delta_{r_1} v(x, t) \frac{\partial}{\partial r_1} \delta_{r_1} v(x, t) - \nu \frac{\partial^2}{\partial x^2} \delta_{r_1} v(x, t) - F(x + r_1, t) + F(x, t) \right] \right\rangle, \end{aligned}$$

where Eq. (49) was used in order to replace the temporal evolution of the velocity increment. Each term can now

be treated separately. Starting with the first advective term we obtain

$$\begin{aligned} & -\frac{\partial}{\partial v_1} \left\langle \delta(v_1 - \delta_{r_1} v(x, t)) v(x, t) \frac{\partial}{\partial x} \delta_{r_1} v(x, t) \right\rangle = \left\langle v(x, t) \frac{\partial}{\partial x} \delta(v_1 - \delta_{r_1} v(x, t)) \right\rangle = \underbrace{\frac{\partial}{\partial x} \langle v(x, t) \delta(v_1 - \delta_{r_1} v(x, t)) \rangle}_{=0, \text{ homogeneity}} \\ & - \left\langle \frac{\partial v(x, t)}{\partial x} \delta(v_1 - \delta_{r_1} v(x, t)) \right\rangle = \int_{-\infty}^{v_1} dv'_1 \frac{\partial}{\partial r_1} \underbrace{\langle \delta(v'_1 - \delta_{r_1} v(x, t)) \rangle}_{=f_1(v'_1, r, t)} - \int_{-\infty}^{v_1} dv'_1 \frac{\partial}{\partial x} \underbrace{\langle \delta(v'_1 - \delta_{r_1} v(x, t)) \rangle}_{=0, \text{ homogeneity}}. \quad (\text{B2}) \\ & \quad \quad \quad = \left[\frac{\partial \delta_{r_1} v(x, t)}{\partial r_1} - \frac{\partial \delta_{r_1} v(x, t)}{\partial x} \right] \times \delta \end{aligned}$$

Here, we made use of the inverse chain rule in the first

and last step. The second advective term can be treated in the same way according to

$$\begin{aligned}
& -\frac{\partial}{\partial v_1} \left\langle \delta(v_1 - \delta_{r_1} v(x, t)) \delta_{r_1} v(x, t) \frac{\partial}{\partial r_1} \delta_{r_1} v(x, t) \right\rangle \\
& = \left\langle \delta_{r_1} v(x, t) \frac{\partial}{\partial r_1} \delta(v_1 - \delta_{r_1} v(x, t)) \right\rangle = \frac{\partial}{\partial r_1} \langle \delta_{r_1} v(x, t) \delta(v_1 - \delta_{r_1} v(x, t)) \rangle - \left\langle \frac{\partial \delta_{r_1} v(x, t)}{\partial r_1} \delta(v_1 - \delta_{r_1} v(x, t)) \right\rangle \\
& = v_1 \frac{\partial}{\partial r_1} \underbrace{\langle \delta(v_1 - \delta_{r_1} v(x, t)) \rangle}_{=f_1(v_1, r_1, t)} + \int_{-\infty}^{v_1} dv'_1 \frac{\partial}{\partial r_1} \underbrace{\langle \delta(v'_1 - \delta_{r_1} v(x, t)) \rangle}_{=f_1(v'_1, r_1, t)}, \tag{B3}
\end{aligned}$$

where we made use of the sifting property of the δ -function, i.e., $\delta_{r_1} v(x, t) \delta(v_1 - \delta_{r_1} v(x, t)) = v_1 \delta(v_1 - \delta_{r_1} v(x, t))$. The nonlinear terms can thus be expressed solely in terms of the one-increment PDF or its associated cumulative PDF which is a particularity of the Burgers equation (for the Navier-Stokes equation we would be

facing unclosed terms from the pressure [54]). However, the viscous contributions in Eq. (B2) confront us with unclosed terms and we have to introduce the two-increment PDF which results in an infinite hierarchy of PDF equations. This can be seen from the following calculation of the viscous term in Eq. (B2)

$$\begin{aligned}
& -\nu \left\langle \delta(v_1 - \delta_{r_1} v(x, t)) \frac{\partial^2 \delta_{r_1} v(x, t)}{\partial x^2} \right\rangle = -\nu \left\langle \delta(v_1 - \delta_{r_1} v(x, t)) \left[\frac{\partial^2 \delta_{r_1} v(x, t)}{\partial r_1^2} - \frac{\partial^2 v(x, t)}{\partial x^2} \right] \right\rangle \\
& = -\nu \int dr_2 [\delta(r_2 - r_1) - \delta(r_2)] \frac{\partial^2}{\partial r_2^2} \langle \delta_{r_2} v(x, t) \delta(v_1 - \delta_{r_1} v(x, t)) \rangle \\
& = -\nu \int dr_2 [\delta(r_2 - r_1) - \delta(r_2)] \frac{\partial^2}{\partial r_2^2} \int dv_2 v_2 \underbrace{\langle \delta(v_2 - \delta_{r_2} v(x, t)) \delta(v_1 - \delta_{r_1} v(x, t)) \rangle}_{=f_2(v_2, r_2; v_1, r_1, t)}. \tag{B4}
\end{aligned}$$

The forcing contributions in Eq. (B2) can be handled by the usual trick of the Langevin equation. Inserting the

above calculations yields the evolution equation for the one-increment PDF

$$\begin{aligned}
& \frac{\partial}{\partial t} f_1(v_1, r_1, t) + v_1 \frac{\partial}{\partial r_1} f_1(v_1, r_1, t) + 2 \int_{-\infty}^{v_1} dv'_1 \frac{\partial}{\partial r_1} f_1(v'_1, r_1, t) \\
& = -\nu \frac{\partial}{\partial v_1} \int dr_2 [\delta(r_2 - r_1) - \delta(r_2)] \frac{\partial^2}{\partial r_2^2} \int dv_2 v_2 f_2(v_2, r_2; v_1, r_1, t) + [\chi(0) - \chi(r_1)] \frac{\partial^2}{\partial v_1^2} f_1(v_1, r_1, t). \tag{B5}
\end{aligned}$$

Appendix C: Reformulation of the viscous term in the multi-increment hierarchy

In this Appendix, we want to show that the unclosed term in the evolution equation of the one-increment PDF (50) involves the local energy dissipation rate. To this end, we rewrite the viscous contributions in their original form according to

$$\begin{aligned}
& \nu \int dr_2 [\delta(r_2 - r_1) - \delta(r_2)] \frac{\partial^2}{\partial r_2^2} \int dv_2 v_2 f_2(v_2, r_2; v_1, r_1) \\
& = \nu \left\langle \left[\frac{\partial^2 \delta_{r_1} v(x)}{\partial r_1^2} - \frac{\partial^2 u(x)}{\partial x^2} \right] \delta(v_1 - \delta_{r_1} v(x)) \right\rangle. \tag{C1}
\end{aligned}$$

A further treatment of these terms yields

$$\begin{aligned}
& +\nu \frac{\partial}{\partial r_1} \left\langle \frac{\partial \delta_{r_1} v(x)}{\partial r_1} \delta(v_1 - \delta_{r_1} v(x)) \right\rangle - \nu \left\langle \frac{\partial \delta_{r_1} v(x)}{\partial r_1} \frac{\partial \delta(v_1 - \delta_{r_1} v(x))}{\partial r_1} \right\rangle \\
& - \nu \frac{\partial}{\partial x} \left\langle \frac{\partial v(x)}{\partial x} \delta(v_1 - \delta_{r_1} v(x)) \right\rangle + \nu \left\langle \frac{\partial v(x)}{\partial x} \frac{\partial \delta(v_1 - \delta_{r_1} v(x))}{\partial x} \right\rangle \\
& \quad \quad \quad = 0, \text{ homogeneity} \\
& = -\nu \int_{-\infty}^{v_1} dv'_1 \frac{\partial^2}{\partial r_1^2} \langle \delta(v'_1 - \delta_{r_1} v(x)) \rangle + \nu \frac{\partial}{\partial v_1} \left\langle \left(\frac{\partial \delta_{r_1} v(x)}{\partial r_1} \right)^2 \delta(v_1 - \delta_{r_1} v(x)) \right\rangle \\
& \quad - \nu \frac{\partial}{\partial v_1} \left\langle \frac{\partial v(x)}{\partial x} \underbrace{\left(\frac{\partial \delta_{r_1} v(x)}{\partial x} \right)}_{= \frac{\partial \delta_{r_1} v(x)}{\partial r_1} - \frac{\partial v(x)}{\partial x}} \delta(v_1 - \delta_{r_1} v(x)) \right\rangle \tag{C2}
\end{aligned}$$

Inserting the one-increment PDF $f_1(v'_1, r_1)$ into the first term on the right hand side yields

$$\begin{aligned}
& = -\nu \int_{-\infty}^{v_1} dv'_1 \frac{\partial^2}{\partial r_1^2} f_1(v'_1, r_1) + \nu \frac{\partial}{\partial v_1} \left\langle \left(\frac{\partial v(x+r_1)}{\partial r_1} \right)^2 \delta(v_1 - \delta_{r_1} v(x)) \right\rangle \\
& \quad + \nu \frac{\partial}{\partial x} \left\langle \underbrace{\left(\frac{\partial v(x)}{\partial x} \right)}_{= \frac{\partial \delta_{r_1} v(x)}{\partial r_1} - \frac{\partial \delta_{r_1} v(x)}{\partial x}} \delta(v_1 - \delta_{r_1} v(x)) \right\rangle + \nu \frac{\partial}{\partial v_1} \left\langle \left(\frac{\partial v(x)}{\partial x} \right)^2 \delta(v_1 - \delta_{r_1} v(x)) \right\rangle \\
& = -\nu \int_{-\infty}^{v_1} dv'_1 \frac{\partial^2}{\partial r_1^2} f_1(v'_1, r_1) + \nu \frac{\partial}{\partial v_1} \left\langle \left(\frac{\partial v(x+r_1)}{\partial r_1} \right)^2 \delta(v_1 - \delta_{r_1} v(x)) \right\rangle \\
& \quad - \nu \int_{-\infty}^{v_1} dv'_1 \frac{\partial^2}{\partial r_1^2} f_1(v'_1, r) + \nu \int_{-\infty}^{v_1} dv'_1 \underbrace{\frac{\partial^2}{\partial r_1 \partial x} f_1(v'_1, r_1)}_{= 0, \text{ homogeneity}} + \nu \frac{\partial}{\partial v_1} \left\langle \left(\frac{\partial v(x)}{\partial x} \right)^2 \delta(v_1 - \delta_{r_1} v(x)) \right\rangle. \tag{C3}
\end{aligned}$$

Under the assumption of homogeneity, we obtain

$$\begin{aligned}
& \left\langle \left(\frac{\partial v(x+r_1)}{\partial r_1} \right)^2 \delta(v_1 - \delta_{r_1} v(x)) \right\rangle \\
& = \left\langle \left(\frac{\partial v(x+r_1)}{\partial r_1} \right)^2 \delta(v_1 - v(x+r_1) + v(x)) \right\rangle \\
& = \left\langle \left(\frac{\partial v(x)}{\partial x} \right)^2 \delta(v_1 - v(x) + v(x-r_1)) \right\rangle \\
& = \left\langle \left(\frac{\partial v(x)}{\partial x} \right)^2 \delta(v_1 + \delta_{-r_1} v(x)) \right\rangle \tag{C4}
\end{aligned}$$

which allows us to introduce the local energy dissipation rate in Eq. (50) according to

$$\begin{aligned}
v_1 \frac{\partial}{\partial r_1} f_1(v_1, r_1) & = 2 \int_{-\infty}^{v_1} dv'_1 \frac{\partial}{\partial r_1} f_1(v', r_1) \\
& \quad - \frac{\partial^2}{\partial v_1^2} \left[\left\langle \frac{\varepsilon(x)}{2} [\delta(v_1 - \delta_{r_1} v(x)) + \delta(v_1 + \delta_{-r_1} v(x))] \right\rangle \right. \\
& \quad \left. + [\chi(0) - \chi(r_1)] f_1(v_1, r_1) \right] + 2\nu \frac{\partial^2}{\partial r_1^2} f_1(v_1, r_1), \tag{C5}
\end{aligned}$$

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