# A minimal phase-coupling model for intermittency in turbulent systems

José-Agustín Arguedas-Leiva,[1] Enda Carroll,[2] Luca Biferale,[3] Michael Wilczek,[1,*] and Miguel D. Bustamante[2]
[1]*Max Planck Institute for Dynamics and Self-Organization, Am Faßberg 17, 37077 Göttingen, Germany*
[2]*School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland*
[3]*Dept. Physics and INFN, University of Rome Tor Vergata, Via Ricerca Scientifica 1, 00133 Rome, Italy*
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Turbulent systems exhibit a remarkable multi-scale complexity, in which spatial structures induce scale-dependent statistics with strong departures from Gaussianity. In Fourier space, this is reflected by pronounced phase synchronization. A quantitative relation between real-space structure, statistics, and phase synchronization is currently missing. Here, we address this problem in the framework of a minimal phase-coupling model, which enables a detailed investigation by means of dynamical systems theory and multi-scale high-resolution simulations. We identify the spectral power-law steepness, which controls the phase coupling, as the control parameter for tuning the non-Gaussian properties of the system. Whereas both very steep and very shallow spectra exhibit close-to-Gaussian statistics, the strongest departures are observed for intermediate slopes comparable to the ones in hydrodynamic and Burgers turbulence. We show that the non-Gaussian regime of the model coincides with a collapse of the dynamical system to a lower-dimensional attractor and the emergence of phase synchronization, thereby establishing a dynamical-systems perspective on turbulent intermittency.



Introduction. Turbulence is a prototypical non-equilibrium phenomenon with a large number of strongly interacting degrees of freedom [1–6]. A salient feature of turbulence is the pronounced scale dependence of statistics with particularly strong departures from Gaussianity observed on the smallest spatial scales. In real space, these departures from Gaussian statistics can be related to coherent, intense and rare, small-scale structures in the gradients of the velocity field – a phenomenon also dubbed as *intermittency* [7, 8]. Departures from Gaussian statistics can also be studied from the complementary perspective of Fourier space. While Gaussian random fields feature completely uncorrelated phases, phase correlations can give rise to complex scale-dependent statistics, and it is well known that coherent spatial structures such as shocks require a high level of correlation amongst the phases of the Fourier modes.

Notably, so far only very few studies have addressed the connection between the emergence of coherent intermittent structures in real space, non-Gaussian statistics and phase correlations, indicating that bursts of spectral energy fluxes (and dissipation) are produced when Fourier phases become correlated [9–11]. Elucidating these connections is important for both fundamental and applied aspects. In particular, we currently miss a clear identification of which dynamical degrees of freedom lead to such bursting and quiescent chaotic alternation of temporal and spatial flow realizations. As a result, we lack optimal protocols to avoid disrupting fluctuations in engineering turbulence [12, 13], to predict extreme events in geophysical flows [14, 15] and to control existence and uniqueness of the PDE solutions [16], just to cite a few main open problems with multidisciplinary impacts. The complexity of fully developed three-dimensional turbulence makes this an extremely challenging task. There is, however, the opportunity to isolate the main aspects of this problem in simpler, more tractable models. One popular way to proceed is to lower the complexity by mode reduction, as in the case of sub-grid-scale modeling [17, 18], Fourier surgery [19, 20], statistical closure [21], partial freezing of some spectral degrees of freedom [22, 23] or asymptotic expansions [24, 25]. All attempts have some merits and deficiencies, the main common drawback being the compromised ability to describe simultaneously spatial and temporal fluctuations on a wide range of scales.

In this paper, we present a minimal description of hydrodynamic turbulence derived from a PDE model, preserving the whole richness of multi-scale spatial and temporal statistics. We combine theory and simulations to establish a clear connection between phase correlations in Fourier space and the complex statistics in real space. The model is formulated in terms of Fourier phases whose dynamical coupling resembles the one in Navier-Stokes turbulence. In essence, its dynamics is reminiscent of Burgers turbulence with the important distinction that in our model, only the phases evolve whereas the amplitudes are kept fixed. This allows us to precisely tune the coupling strength of the phases by controlling the slope of the energy spectrum.

We find that the system transitions from Gaussian to non-Gaussian statistics as the spectrum is gradually steepened. For slopes beyond a certain value, the rare fluctuations become less extreme and return to near-Gaussian statistics. Strikingly, the strongest deviations occur in the intermediate range, within the range of values attained by turbulent systems. Within this range, the dimension of the strange attractor collapses to a minimum, indicating that non-Gaussian real-space statistics are related to the collapse of the dynamical system onto a lower-dimensional manifold.

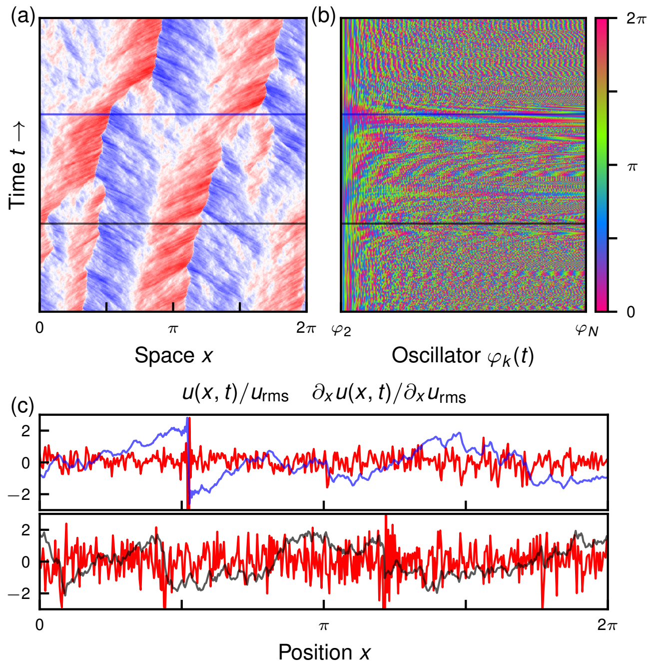


FIG. 1. Results of a numerical simulation of equations (6) with $k_0 = 1$ and $N = 2^9$ degrees of freedom, with a Burgers-like steepness parameter $\alpha = 1$. Panel (a): Color plot of reconstructed physical field $u(t,x)$ displaying a shock near $x = \pi/2$. Panel (b): Corresponding color plot of individual phases $\varphi_2, \ldots \varphi_N$. In both panels, the gray line marks an early snapshot illustrating an instance of a disordered regime while the blue line marks a late snapshot illustrating a persistent synchronization regime. Panel (c): for the disordered (bottom, gray) and synchronized (top, blue) regimes, the corresponding snapshots of the reconstructed physical field $u(t,x)$. Red plots show the gradient $\partial u(t,x)/\partial x$ to illustrate the difference between the two regimes.

Our work sheds light on the emergence of coherent structures and the associated phase synchronization phenomena [26], establishing deep connections between the statistical theory of non-equilibrium systems and dynamical systems theory.

The model. As a starting point, let us consider the one-dimensional Burgers equation

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = \nu \partial_x^2 u(t,x). \quad (1)$$

This simple prototypical equation possesses a quadratic nonlinearity reminiscent of the one in the Navier-Stokes equations, and it is known to develop multi-scale bifractal scaling properties, shocks, non-Gaussian statistics and many other non-trivial statistical features [27–36]. Hence it has become a workhorse in turbulence theory to investigate the effects of nonlinear advection. We consider a one-dimensional field $u(t,x)$ on a $2\pi$-periodic domain with Fourier decomposition

$$u(t,x) = \sum_k a_k(t)\, e^{\mathrm{i}(\phi_k(t)+kx)}. \quad (2)$$

By inserting the Fourier representation into the Burgers equation (1), we obtain equations of motion for the amplitudes and the phases

$$a_k \frac{\mathrm{d}\phi_k}{\mathrm{d}t} = \sum_p -k\, a_p\, a_{k-p}\, \cos(\phi_p + \phi_{k-p} - \phi_k), \quad (3)$$

$$\frac{\mathrm{d}a_k}{\mathrm{d}t} = \sum_p k\, a_p\, a_{k-p}\, \sin(\phi_p + \phi_{k-p} - \phi_k) - \nu k^2 a_k. \quad (4)$$

This infinite set of coupled ODEs describes the full Burgers dynamics of the Fourier phases and amplitudes. Recently, it was shown that the dynamics of the Fourier phases $\phi_k(t)$ determine to a great extent the shock dynamics and the associated non-Gaussian statistics [9, 11]. Thus, we take equation (3) as a starting point for a minimal model for Fourier phase dynamics in turbulence, the "phase-only" model. We set the amplitudes to prescribed constants

$$a_k = |k|^{-\alpha},\ |k| > k_0, \qquad a_k = 0,\ |k| \le k_0, \quad (5)$$

where the steepness $\alpha$ is a continuous control parameter and $k_0 > 0$ is introduced as a large-scale cutoff leading to a finite integral length scale. We find that this destabilises a single-shock-like fixed point, allowing for non-steady dynamics. The phase dynamics is obtained from equation (3) which becomes a system of coupled oscillators $\varphi_k$ satisfying

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}t} = \sum_p \omega_{k,p}\, \cos(\varphi_p + \varphi_{k-p} - \varphi_k), \quad |k| > k_0\,, \quad (6)$$

with coefficients $\omega_{k,p} = -k\ |p(k-p)|^{-\alpha}\ |k|^{\alpha}$ when $|p-k| > k_0$ and $|p| > k_0$ ($\omega_{k,p} = 0$ otherwise), and with $\varphi_{-k} = -\varphi_k$ (reality condition). This triadic interaction term couples the phases with wavenumbers $k$, $p$, and $k-p$, via the so-called triad phase $\varphi^k_{p,k-p} := \varphi_p + \varphi_{k-p} - \varphi_k$. It is important to note that this phase-only model does not need an energy input/output mechanism, as constant energy is maintained by the constant amplitudes. Furthermore, it is formally fully time reversible under the symmetry $t \to -t; \varphi_k \to \varphi_k + \pi$. However, it will not come as a surprise that, like in a formally reversible version of the Navier-Stokes equations [37–40], the chaotic dynamics spontaneously break the time symmetry leading to a non-Gaussian and skewed velocity probability distribution function (PDF). To study the model numerically, we further introduce a discretization with grid spacing $\Delta x = 2\pi/N$. The reality condition $\varphi_{-k} = -\varphi_k$ leaves us with a set of phases evolving on modes $k_0 < k \le N-1$. We set $k_0 = 1$ so $a_1 = a_{-1} = 0$ and thus the evolving variables are $\varphi_2, \ldots, \varphi_{N-1}$. Note that the energy spectrum of the field is fixed and perfectly self-similar: $E_k \sim a_k^2$, with a power-law decay of $E_k \propto k^{-2\alpha}$. The observed original Burgers' case, where quasi discontinuities (shocks) dominate the high-order statistics, corresponds to $\alpha = 1$.

Numerical results on real-space and phase dynamics & statistics. We integrate numerically (6) with the Runge-Kutta method starting from uniformly random initial conditions. The nonlinear term can be written as a convolution, which we efficiently evaluate with a pseudospectral method. Figure 1 illustrates the dynamics of our model, for the choice of steepness $\alpha = 1$ (Burgers shock case), revealing insights into the relation between non-Gaussianity of the real-space statistics and Fourier phase synchronization. Panel (a) is a space-time plot of the velocity field from this minimal model, showing that shocks are the dominant structure. As time evolves, shocks steadily merge and separate. Occasionally, they merge into one dominating shock (horizontal blue line). Panel (b) is a time plot of the individual Fourier phases of the model. It shows that the presence of this dominating shock is due to synchronization of the oscillator model. Away from these synchronization bursts, the system is dominated by smaller shocks and we observe only little coherence (gray line). Panel (c) shows that the synchronization events and presence of the dominating shock (blue line) yield extreme events in the gradient field characterizing the small scales of the velocity field.

By changing the free parameter $\alpha$ in the phase model (6) we can control the multi-scale properties of the coupling of the phases which, in turn, has a direct influence on the hierarchical organization of typical time scales. In a *local* approximation, i.e. supposing the dynamics at wavenumber $k$ is mainly driven by triads around the same wavenumber, $|k| \sim |k-p| \sim |p|$, we can estimate the scale-dependent eddy-turnover time as $\tau_k \sim |k|^{\alpha-1}$, indicating that within this approximation we reach a regime where small spatial scales are faster than the large spatial scales if $\alpha < 1$ (and slower if $\alpha > 1$). The local-triad approximation is expected to be valid in the range $0.5 < \alpha < 1.5$ [41], where the Fourier transform connecting spectrum and two-point velocity correlations does not diverge neither in the UV nor in the IR. As a result, we expect that in the above range and around $\alpha = 1$ a non-trivial balancing between spatial and temporal fluctuations will set in.

In figure 2 we indeed observe that our model has a non-trivial scale and steepness-dependent statistics. Here we show the probability distribution functions (PDF) of the velocity increments $\delta_r u = u(x+r) - u(x)$ for two different values, $r = L \equiv \pi$ and $r = \eta \equiv \pi/N$ denote the largest and smallest distances in the periodic domain, respectively. Real-space statistics are obtained by inserting the phase dynamics into the Fourier representation (2). For completely uniform Fourier amplitudes (steepness $\alpha = 0$) the phases evolve under an all-to-all coupling with equal coupling strength. Note that this choice of spectral amplitudes corresponds to a delta-correlated field in real space. In this case, all phases become dynamically uniformly distributed and uncorrelated. In real space this produces a Gaussian velocity field at all scales (panel (a) in figure 2). In contrast, for steepness values within the range $[0.5, 1.5]$, where the local-triad approximation is expected to be valid, heavy tails are observed in the PDF of the small scales of the velocity field (panels (b)-(d) in figure 2). For the smallest increment PDF, the negative tails are much heavier than the positive tails and both are much heavier than Gaussian. Heuristically (to be quantified later), this is the result of phase synchronization leading to shocks (anti-shocks), i.e. extreme negative (positive) events at the smallest scales.

The presence of extreme events is maximal at $\alpha \sim 1.25$, as evidenced in figure 2(c). For higher values of $\alpha$ the PDF tails slowly regularize. In this limit, the large-scale modes dominate the real-space velocity field. This leads to a dominant sinusoidal mode with superimposed smaller fluctuations. As a consequence the large $\alpha$ limit shows close to Gaussian statistics throughout.

To quantify the steepness-dependent departure of the small scales from Gaussianity we measure the skewness and flatness at different increments $r$:

$$S(r) = \frac{\langle (\delta_r u)^3 \rangle}{\langle (\delta_r u)^2 \rangle^{3/2}}; \qquad F(r) = \frac{\langle (\delta_r u)^4 \rangle}{\langle (\delta_r u)^2 \rangle^2}. \tag{7}$$

Note that due to our frozen-amplitude conditions the denominators of both quantities do not fluctuate. In figure 3(a) we observe a clear transition at $\alpha \sim 0.75$. The peaks of skewness and flatness at $\alpha \sim 1.28$ correspond to the presence of extremely intense negative gradients seen in figure 2(c).

As the steepness is increased further, the phases evolve under a non-local and non-trivial triad coupling. This gives rise to synchronization events, which underlie the steepness-dependent transition observed in real-space statistics. Note, however, that when steepness is too large the timescales from the triad coupling can get too separated, as the coefficients $\omega_{k,p}$ in (6) become too small when $|p|$ and $|k-p|$ are large. Thus we expect to see synchronization over a finite range of steepness values only. In the next sections we will quantify the dependence, on the $\alpha$ parameter, of synchronization and of the structure of the associated chaotic attractors.

Synchronization. In order to quantify the behaviour of triad phases across a range of scales for the oscillator system (6), we define the scale-dependent collective phase $\theta_k$ via:

$$e^{i\theta_k} = i \frac{\sum_p a_p a_{k-p} e^{i(\varphi_p + \varphi_{k-p} - \varphi_k)}}{\left| \sum_p a_p a_{k-p} e^{i(\varphi_p + \varphi_{k-p} - \varphi_k)} \right|}. \tag{8}$$

This collective phase is dynamically relevant as the RHS of the evolution equations (6) is proportional to $\sin \theta_k$. The fluctuations of $\theta_k$ over time serve as a measure of the triad phase coherence across scales. Thus, averaging over time we get the following scale-dependent Kuramoto

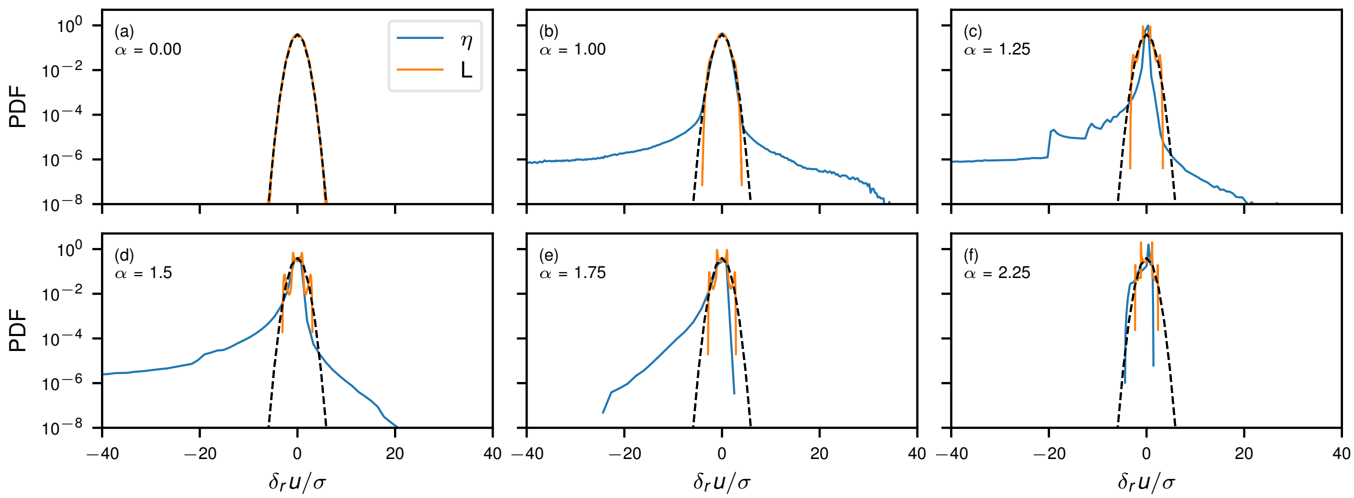


FIG. 2. (a)-(f): Standardized PDFs of $\delta_r u$ calculated at the the smallest, $\eta$, and largest increments, $L$. (a) For completely flat Fourier amplitudes $\alpha = 0$ the velocity field is Gaussian across all scales. (b)-(e) Increasing $\alpha$ leads to heavy tails at small scales. This indicates that extreme events are more common at small scales than at large scales, which in turn remain closer to Gaussian throughout. (f) For a steep enough spectrum the velocity field is dominated by the first few modes. This regularizes the heavy tails. These histograms correspond to simulations with $N = 2^{15}$ collocation points and large-scale cutoff $k_0 = 1$.

order parameter:

$$R_k e^{i\Theta_k} = \langle e^{i\theta_k} \rangle_t \tag{9}$$

As usual we have $0 \leq R_k \leq 1$ and phase synchronization is indicated by $R_k$ values close to 1. Averaging additionally over the spatial scales, we define the average phase synchronization by

$$\mathcal{R}(\alpha) = \frac{1}{N - k_0} \sum_{k=k_0+1}^{N} R_k, \tag{10}$$

which measures how the phase synchronization changes as a function of the spectral slope.

Figure 3(b) shows the average phase synchronization as a function of steepness $\alpha$ for various system sizes $N$. The relatively high synchronization seen for small system size at $\alpha > 2.0$ decreases as the system size is increased. This is due to the addition of faster and noisier oscillators to the system causing a convergence towards a pronounced peak for $\alpha \in [1.0, 2.0]$, indicating high phase synchronization for this interval for large system sizes. This synchronization peak is remarkably coincidental with the flatness and skewness peaks shown in figure 3(a). This provides quantitative evidence in support of the relation between synchronization (a dynamical-system measure) and intermittency (a real-space measure). Chaos characterization. As an additional characterization of the dynamical system, we estimate the properties of the underlying strange attractor as a function of $\alpha$ and for system sizes $N = 64, 128, 256, 512$ by examining the Lyapunov exponents (LEs) [42]. For reasons of numerical complexity we cannot reach the same resolution we used for the statistical characterisation of intermittency and synchronization; however as we will see below the $N = 512$ case shows strong indications of convergence to the large-$N$ limit.

Using the LEs we can calculate the dimension of the attractor via the Kaplan-Yorke approximation [43, 44]. Given the ordered LEs $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1}$, the Kaplan-Yorke dimension is defined as

$$\mathcal{D}_{KY} = i + \frac{\sum_{j=1}^{i} \lambda_j}{|\lambda_{i+1}|}, \tag{11}$$

where the conditions $\sum_{j=1}^{i} \lambda_j \geq 0$ and $\sum_{j=1}^{i+1} \lambda_j < 0$ define the index $i$. The Kaplan-Yorke dimension gives a measure of the systems' effective degrees of freedom. Figure 3(c) shows a plot of the ratio between the Kaplan-Yorke dimension and the number of available degrees of freedom, as a function of $\alpha$ and for several values of the system size $N$. It is evident that as $N$ grows a clear pattern emerges, whereby the Kaplan-Yorke dimension greatly diminishes for values of $\alpha$ inside the interval $[1.0, 2.0]$, a behaviour that coincides, on the one hand, with the departure from Gaussianity observed in figure 3(a), and on the other hand, with the increase in phase synchronization shown in figure 3(b).

Conclusions. Our minimal model sheds light into the nature of coherent structures as low-dimensional objects, as it establishes a dynamical scenario where real-space intermittency and phase synchronization are accompanied by a reduction in the dimensions of the strange attractor. Remarkably, this model's coherent structures are controlled by Fourier phase dynamics only, as the energy spectrum is static and plays a background role only, via interaction coefficients.

Our results open new perspectives concerning the possibility to connect turbulence intermittency with dynam-

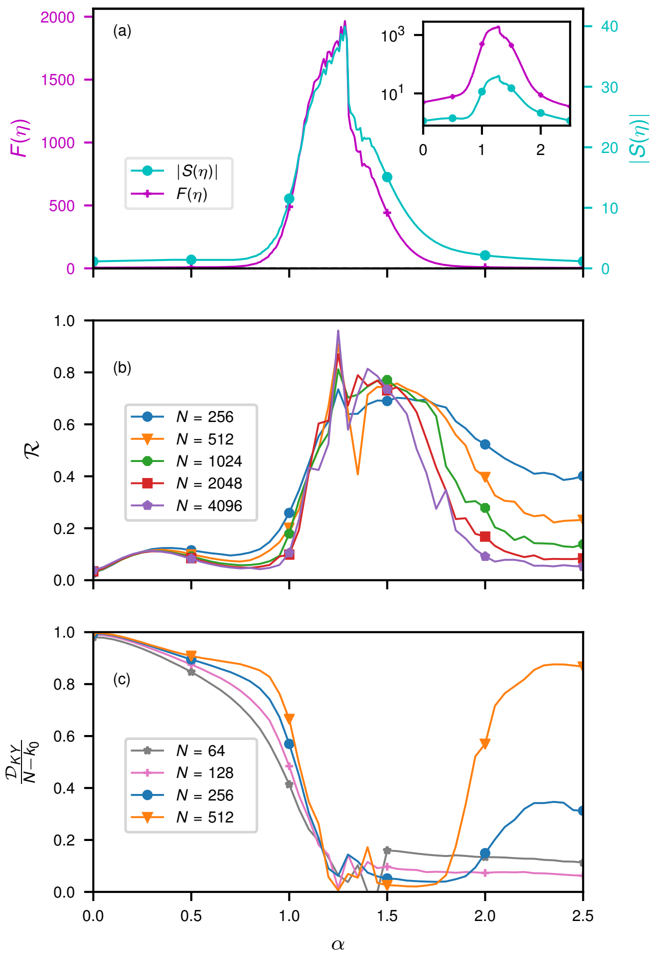


FIG. 3. Panel (a) shows the absolute value of the skewness $|S(\eta)|$ and the flatness $F(\eta)$ of the PDF of the velocity increments defined in equation (7) for the smallest increment $\eta$ as a function of steepness $\alpha$ (inset: same figure on log-lin scales). Panel (b) shows the average phase synchronization, equation (10), as a function of $\alpha$ for various system sizes. Panel (c) shows the ratio between the Kaplan-Yorke dimension, equation (11), and the available degrees of freedom as a function of $\alpha$ for various system sizes.

ical system tools based on phase synchronization and chimera states [45].

On the quantitative side, our results provide insight on the solution to the full inviscid Burgers equation, where all amplitudes are allowed to evolve. There, for generic initial conditions, a finite-time singularity develops characterised by phase synchronization and a power-law spectrum with steepness $\alpha \in [1.33, 1.50]$ [46, 47]. The same behaviour is observed even if we impose the constraint $a_{k_0} = 0$ for $k_0 = 1$. Because in the full equations the spectrum evolves slowly, it is natural to expect that in our constrained model, with frozen spectrum, the phases must show high correlation in the same range of imposed slopes.

A natural extension of this work would be an investigation of the phase-only 3D Navier-Stokes dynamics by fixing the amplitudes of all Fourier modes, including comparisons to Navier-Stokes equations with a fixed spectrum, either for all wavenumbers or for a subset of them [48, 49]. Results in this direction would help to shed additional light on the origin of extreme events and small-scale intermittency.

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* michael.wilczek@ds.mpg.de

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