# Multifractality in the Statistics of the Velocity Gradients in Turbulence

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Using the multifractal approach, we derive the probability distribution function (PDF) of the velocity gradients in fully developed turbulence. The PDF is given by a nontrivial superposition of stretched exponentials, corresponding to the various singularity exponents. The form of the distribution is explicitly dependent on the Reynolds number. The experimental data are in good agreement with the PDF predicted by the same random beta model used to fit the scaling of the velocity structure functions.

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One of the fundamental features of three-dimensional fully developed turbulence is the non-Gaussian statistics at small scales. The energy transfer toward small scales is related to the nonzero skewness of the probability distribution function (PDF) of the gradients, and the large flatness of the PDF (kurtosis) corresponds to the presence of strong bursts in the energy dissipation. This is the most striking signature of the so-called intermittency phenomenon, responsible for the failure of the classical theory of Kolmogorov (K41) which neglects the presence of fluctuations in the energy transfer [1].

Recently, several papers [2–5] have discussed the PDF problem. A first approach [2–4] uses a new mapping-closure theory introduced by Kraichnan [2]. Starting from a Gaussian reference field, he introduced a mapping function $J$ describing the squeezing ratio of the length scale. The evolution equation for $J$ is obtained by modeling the dynamical processes present in the Navier-Stokes equations with a particular emphasis on the local self-distortion of turbulent structures in physical space. A second method applied dimensional arguments in order to relate the small-scale fluctuations to large-scale statistics assumed to be Gaussian [5]. Thus, explicit forms of the PDF are derived in the context of the K41 theory and of the fractal beta model [6,7]. However, these forms are not consistent with the existing data.

This Letter generalizes the approach used in Ref. [5] to the multifractal [6,8,9] case. In particular, we show that the random beta model [9] allows us to obtain good fits of both the scaling exponents of the structure functions and the PDF of the gradients.

Let us briefly recall the basic features of the multifractal description. In the following, we shall ignore the vectorial character of the quantities, as well as constants of order 1 in the equalities, which are unessential for our dimensional arguments.

In the inertial range of lengths, the velocity increments $\delta v_x(l) = |v(x+l) - v(x)|$ are assumed to scale as

$$\delta v_x(l) \propto v_0 l^h , \tag{1}$$

where $v_0 = |V_0|$ is the absolute value of the characteristic velocity difference $V_0$ on the typical macroscopical length $L_0$. For the sake of simplicity, we assume $L_0 = 1$. In general, the scaling exponent $h$ depends on the particular fluid point $x$ considered. Fluctuations of $h$ lead to a strong spatial intermittency in the magnitude of the gradients and thus in the energy dissipation. The main assumption of the fractal description of intermittency [6] is that $h < 1$ in a set $F$ with fractal dimension $D_F < 3$ where the energy dissipation concentrates as the Reynolds number $\mathrm{Re} \to \infty$. The complement of this set is covered by regular regions (which are not fractal) where the velocity field can be linearized, i.e., $h \geq 1$, so that the gradients remain small at high Re. $F$ is a multifractal [8] if considered as a superposition of subsets $\Omega(h)$ of points $x$ such that $\delta v_x \sim l^z$ with $z \in [h, h+dh]$. Multifractality is characterized by a function $D(h)$ which is the fractal dimension of the set $\Omega(h)$. We remark that in the multifractal description $D_F = 3$ is also allowed. In practice, one extracts $D(h)$ from the scaling exponents $\zeta_p$ of the structure functions $\langle \delta v^p \rangle \sim l^{\zeta_p}$. In fact, one can argue that the probability of picking up a singularity exponent $h$ should scale in the inertial range as the fraction of the coarse graining measure of the volume of $\Omega(h)$ over the total volume, that is,

$$\mathcal{P}_l(h) dh = l^{3-D(h)} \rho(h) dh , \tag{2}$$

where $\rho(h)$ is a smooth function of $h$ which is independent of $l$. The scaling ansatz (2) implies that

$$\zeta_p = \min_h [hp + 3 - D(h)] , \tag{3}$$

using a saddle-point estimate of the structure functions [8].

 

It is important to have in mind that the velocity increments are self-similar only in the inertial range up to a length scale $l_D$ where viscous effects become comparable to nonlinear transfer. $l_D$ is thus determined by imposing that the effective Reynolds number on scale $l$ is equal to unity, i.e.,

$$\delta v(l_D) l_D/\nu = 1 . \qquad (4)$$

It follows [10] that the dissipative scale $l_D$ is itself a function of $h$,

$$l_D(h) \sim (\nu/v_0)^{1/(1+h)} . \qquad (5)$$

The stronger the singularity exponent, the smaller the value of the corresponding dissipative scale $l_D(h)$. In order to stop the cascade one has to require that the smallest singularity exponent $h_{\min} > -1$. Because of incompressibility constraints [11], the value $h_{\min}=0$, however, seems more reasonable and it is consistent with the present experimental data [12,13]. The velocity gradients $s$ can be expressed in terms of the singularities $h$ via (1) and (5) as

$$|s| \approx \delta v(l_D)/l_D = v_0 l_D^{h-1} = v_0^{2/(1+h)} \nu^{(h-1)/(h+1)} . \qquad (6)$$

In order to determine its PDF, one can relate the probability of the velocity increments on small scales to the probability $\Pi(V_0)$ of the characteristic velocity difference $V_0$ on large scales. $\Pi(V_0)$ is usually assumed to be Gaussian on the basis of the experimental results as well as of heuristic central-limit arguments [1]. For our purpose, it is convenient to consider the conditional probability $P_h(s)$, i.e., the PDF restricted to points belonging to the subset $\Omega(h)$ characterized by a particular singularity exponent [5]. Using the relation

$$P_h(s) = \Pi(V_0) \left| \frac{dV_0}{ds} \right| , \qquad (7)$$

and expressing $V_0$ in terms of $s$ via (6), it follows that

$$P_h(s) \sim \left( \frac{\nu}{|s|} \right)^{(1-h)/2} \exp\left( -\frac{\nu^{1-h}|s|^{1+h}}{2\langle V_0^2 \rangle} \right) . \qquad (8)$$

The prediction of the K41 theory is obtained by neglecting intermittency effects, i.e., by assuming $h=\frac{1}{3}$ uniformly in the fluid. One thus finds [5]

$$P(s) \sim (\nu/|s|)^{1/3} \exp[-C\nu^{2/3}|s|^{4/3}] , \qquad (9)$$

with $C=(2\langle V_0^2\rangle)^{-1}$. Let us recall that in a fractal picture, the velocity gradients are very small in the nonactive zones where $h \geq 1$. Therefore, we must take into account the presence of an additional delta function in the PDF:

$$P(s) = \bar{P}_F(s) + \gamma\delta(s) , \qquad (10)$$

where $\bar{P}_F$ is the PDF restricted to the active zones covering the fractal set $F$ and $\gamma$ is a normalization factor, such that $\int P(s) ds = 1$. For instance [5], $\bar{P}_F(s) = P_{\bar{h}}$ in the beta model where the singularity value on $F$ is constant and equal to $\bar{h} = (D_F - 2)/3$. It is important to stress that both the K41 theory and the beta model predict a PDF with stretched exponentials of the form $\exp(-c|s|^t)$, with $t>1$. In a log-linear plot of the PDF, such a form implies a convexity of the curve which is in contradiction with the qualitative feature of the experimental and numerical data [14,15]. Indeed, as one can see in Fig. 2, they seem consistent with an effective stretched exponent $t<1$.

Let us now derive the form of the PDF in the multifractal approach. As a matter of fact, when there is a hierarchy of singularities, the probability of observing a gradient value $s$ related to a given singularity $h$ is $P_h(s)\mathcal{P}_{l_D}(h)$, where $\mathcal{P}_l$ is given by (2). It follows that the conditional probability is given by a weighted integral over the singularities,

$$\bar{P}_F(s) = \int dh\, P_h(s) \mathcal{P}_{l_D}(h) \sim \int dh\, \rho(h) \left( \frac{\nu}{|s|} \right)^{2-[h+D(h)]/2} \exp\left( -\frac{\nu^{1-h}|s|^{1+h}}{2\langle V_0^2\rangle} \right) . \qquad (11)$$

An analytic estimate of the integral is not easy, since one cannot apply a saddle-point method as for the analogous integral giving the structure functions. This is due to the fact that, as shown by (6), $|s| \sim \nu^{(h-1)/(h+1)}$. Even for strong singularities $h \approx 0$, one has $|s| \sim \nu^{-1}$, and we therefore expect that $|s|\nu$ should be of order unity in the viscosity. We remark that in the multifractal model the relation $\nu\langle s^2\rangle \rightarrow$ const in the limit of Re $\rightarrow \infty$ immediately follows [16] from the condition $\zeta_3 = 1$.

It is of course possible for a numerical computation of the integral, inserting the dimension function $D(h)$ obtained by the Legendre transformation (3). In order to have an explicit form of $D(h)$, we can use the random beta model [9]. It is one of the simplest multifractal models which provides a good fit of structure functions. We show here that it is also capable to reproduce the gradients statistics with the same accuracy.

In the random beta model [9] the velocity difference $v_n \equiv \delta v(l_n)$ on the scale $l_n = 2^{-n}$ is given by

$$v_n = v_0 l_n^{1/3} \prod_{i=1}^{n} \beta_i^{-1/3} , \qquad (12)$$

where the $\beta_i$'s are independent, identically distributed random variables. Phenomenological arguments suggest the following choice: $\beta_i = 1$ with probability $\alpha$ or $\beta_i = B \equiv 2^{-(1-3h_{\min})}$ with probability $1-\alpha$. The two limit cases are $\alpha=1$ corresponding to the K41 theory and $\alpha=0$ corresponding to the usual beta model [7], where at the end of the cascade the energy concentrates on a fractal structure with dimension $D_F = 2+3h_{\min}$. A fit of the experimental data of Anselmet *et al.* and the more recent data of Meneveau and Sreenivasan [12] for the $\zeta_p$ exponents gives the value $\alpha=\frac{7}{8}$, assuming $h_{\min}=0$, i.e., $B=\frac{1}{2}$.

Instead of inserting in (9) the corresponding function $D(h)$ given by the random beta model [10], one can obtain a simpler formula for the PDF, by a direct estimation. The results are of course equivalent, although the latter permits a more transparent reading. From (12), one sees that the probability distribution of the velocity increments is

$$P(v_n)=\int \Pi(V_0)dV_0\int \delta\left[v_n-v_0 l_n^{1/3}\prod_{i=1}^{n}\beta_i^{-1/3}\right]\prod_{i=1}^{n}\beta_i\mu(\beta_i)d\beta_i\,, \tag{13}$$

where $\mu(\beta)$ is the probability density of the $\beta_i$'s. Since the $\beta_i$'s are identically distributed according to a binomial distribution, the integral can be reduced to the sum

$$P(v_n)\sim\sum_{K=0}^{n}\binom{n}{K}\alpha^{n-K}(1-\alpha)^K(B)^{4K/3}l_n^{-1/3}\exp[-CB^{2K/3}l_n^{-2/3}v_n^2]\,, \tag{14}$$

where $C=(2\langle V_0^2\rangle)^{-1}$ and $l_n=2^{-n}$. Figure 1 shows $P(v_n)$ for different values of $n$. The passage of the velocity increments PDF from a Gaussian form at large scales to an exponential-like form at small scales is quite evident.

The gradient PDF is obtained by (14), computed at the step $N$ which corresponds to the viscous cutoff (5), $v_N l_N/\nu=1$, namely,

$$l_N^2\equiv 2^{-2N}\sim \nu/s\,.$$

Because the cascade stops at $N=\ln(s/\nu)/(2\ln 2)$, it follows that $B^{2N}=(\nu/s)^{1-3h_{\min}}$ and the conditional PDF is

$$\bar{P}_F(s)=\sum_{K=0}^{N}\binom{N}{K}\alpha^{N-K}(1-\alpha)^K\left(\frac{\nu}{|s|}\right)^{(1+2k)/3}\exp[-C\nu^{(2+k)/3}|s|^{(4-k)/3}]\,, \tag{15}$$

where $k=K(1-3h_{\min})/N$ and $N=\ln(|s|\nu)/(2\ln 2)$. The K41 prediction (9) corresponds to considering only the term $K=0$, with $\alpha=1$. When $N(s)$ is not large, the main contribution to the sum is given by the first $K$ terms. For increasing $|s|$, the PDF becomes sensitive to higher $K$ terms, i.e., to stronger singularities. A direct inspection of (15) shows that the largest contributions to the sum are given by $K$ around a $K^*(s)$ which exhibits a very weak dependence on $s$, probably logarithmic. The absence of any *dominant* contribution implies that one never has a pure exponential (eventually stretched) form of the PDF. The maximal value of $N(s)$ is limited by the Reynolds number of the experiment: Typical values are $N=14$–$16$. Figure 2 shows the comparison between the numerical data of Vincent and Meneguzzi [15] and Eq. (15), with the same parameters, $h_{\min}=0$ and $\alpha=\frac{7}{8}$, used for the fit [9,10] of the structure function exponents $\zeta_p$.

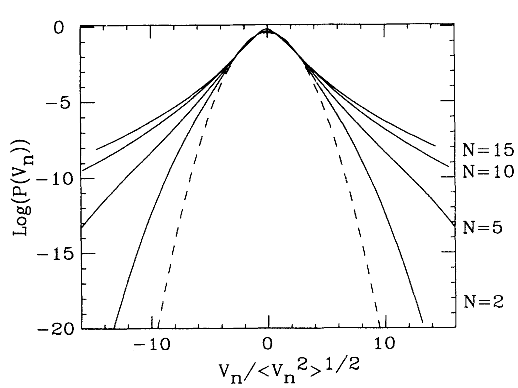


FIG. 1. Log-linear plot of the probability distribution $P(v_n)$ of the velocity increments $v_n$ vs $v_n/\sigma_n$, where $\sigma_n^2=\langle v_n^2\rangle$, for $n=2$, 5, 10, 15 (solid lines); see Eq. (14) with $\alpha=\frac{7}{8}$, $\langle V_0^2\rangle=1$. For $n=0$ one has a Gaussian corresponding to a parabola (dashed line).

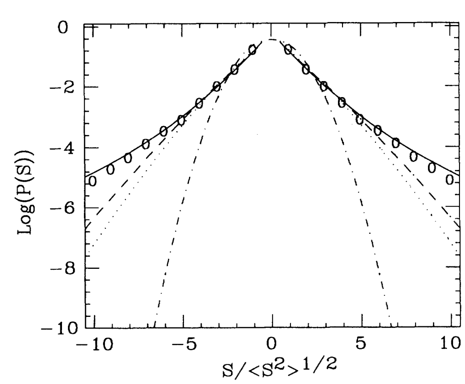


FIG. 2. Log-linear plot of the PDF of the gradients $\bar{P}_F(s)$ vs $s/\sigma$, where $\sigma^2=\langle s^2\rangle$. We choose $\langle V_0^2\rangle=1$, $\nu=10^{-3}$, and $L_0=1$ to be consistent with the simulation [15]. The data of Ref. [15] are indicated by circles; the solid line is the multifractal prediction of Eqs. (11) or (15) with $\alpha=\frac{7}{8}$; the K41 prediction (9) and the beta model result (8) with $D_F=2.83$ ($h\sim 0.27$) are, respectively, the dotted and the dashed lines.

Note that the $\zeta_p$'s obtained in Ref. [15] are in good agreement with the experimental results [11] considered in Refs. [9] and [10].

Equation (11) can be immediately specialized to the Burgers equation. Apart from the regular regions, there is only one kind of singularity $h=0$ with $D(h=0)=0$, corresponding to the presence of shocks. The regular zones have $h=1$ and $D(h=1)=1$ so that the scaling exponents of the structure functions are $\zeta_p=p$ for $p\leq 1$ and $\zeta_p=1$ for $p\geq 1$. As we are dealing with a one-dimensional system, one has $\mathcal{P}_l(h)\sim l^{1-D(h)}$ instead of (2). The integral (11) for the PDF is then given by just one term:

$$\bar{P}_F(s)\sim(\nu/|s|)\exp[-C\nu|s|]\,. \quad (16)$$

This result is valid for large negative $s$ because the strong jumps in the velocity increments are now only negative. Equation (16) coincides with the result obtained by Kraichnan [2] using the mapping-closure approach.

In conclusion, we have obtained the predictions of the multifractal theory for the PDF of the velocity gradients, in both the Navier-Stokes and Burgers equations. In the Navier-Stokes case, the PDF has no stretched or pure exponential form, but is a superposition of stretched exponentials, each one linked to a singularity exponent $h$. The global—but deceptive—effect of the slow increasing with $s$ of $K^*(s)$ is not too different from a stretched exponential $\exp(-c|s|^t)$ with an exponent $t$ slightly less than 1, as shown in Fig. 2. We argue that such a feature suggested misleading interpretations of the experimental results. Here, we have obtained a good agreement with the available data using the random beta model with the same value of the free parameters ($\alpha=\frac{7}{8}$, $h_{\min}=0$) given by an independent previous fit of the structure functions [9].

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