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Multi-time, multi-scale correlation functions in turbulence and in turbulent models

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Abstract

A multifractal-like representation for multi-time, multi-scale velocity correlation in turbulence and dynamical turbulent models is proposed. The importance of subleading contributions to time correlations is highlighted. The fulfillment of the dynamical constraints due to the equations of motion is thoroughly discussed. The predictions stemming from this representation are tested within the framework of shell models for turbulence. ©1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Turbulent flows are characterized by a highly chaotic and intermittent transfer of velocity fluctuations from the stirring length, *outer* scale, L_0, down to the viscous dissipation length, *inner* scale, l_d. The Reynolds number defines the ratio between the outer and the inner scales: $L_0/l_d = Re^{3/4}$. We talk about fully developed turbulent flows in the limit $Re \to \infty$, in this limit it is safely assumed that there exists an inertial range of scales, $l_d \ll r \ll L_0$, where the dynamics is dominated by the inertial terms of the Navier–Stokes equations.

The highly chaotic and intermittent transfer of energy leads to non-trivial correlation among fluctuations of the velocity fields at different scales and at different time-delays [1].

The natural set of observable which one would like to control are the following:

$$C^{p,q}(r, R|t) = \langle \delta v_r^p(t) \cdot \delta v_R^q(0) \rangle \qquad (1)$$

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where $\delta v_r(t) = v(x+r,t) - v(x,t)$ and $l_d \ll r < R \ll L_0$. In (1) we have, for the sake of simplicity, neglected the vectorial and tensorial dependencies in the velocity fields and velocity correlations, respectively.

It is important to point out that the time dependence of correlations as (1) is trivial whenever the velocity difference is computed in the laboratory frame of reference. In this case, the major effect is due to the sweeping of small-scale eddies by large-scale ones, which leads to correlation times scaling as $R/\delta v_{L_0}$. The behavior of time correlations in the laboratory frame bears thus no relation to the "true" dynamical time-scale, $\tau_R \sim R/\delta v_R$, which is associated to the energy transfer. To bypass this problem one has to get rid of the sweeping: this can be accomplished by moving to a reference frame attached to a parcel of fluid in motion, the Quasi-Lagrangian frame of reference [2]. It can be shown that single-time correlations of velocity differences in the Quasi-Lagrangian reference frame are the same as those in the laboratory frame, as a consequence of statistical time stationarity. Hereafter, we shall always refer to velocity differences in the Quasi-Lagrangian frame.

As a consequence of having eliminated the sweeping effects from the many time statistics, every phenomenological description that applies to fully developed Navier–Stokes turbulence can be straightforwardly abridged to make it applicable to shell models of turbulence (for a recent review see [11]). Shell models in this sense can be viewed as a shorthand of Navier–Stokes equations in the Quasi-Lagrangian frame of reference.

Some subclasses of the multi-scale, multi-time correlation functions (1) have lately attracted the attention of many scientists [2–4,6,8,10]. By evaluating (1) with $r = R$, at changing R, and at zero-time delay, $t = 0$, we have the celebrated structure functions of order $p+q$. Further, we may also investigate multi-scale correlation functions when we have different lengths involved $r \neq R$ at zero delay, $t = 0$ as well as single-scale correlation functions by fixing $r = R$ at varying time-delay t etc.

Structure functions have been, so far, the most studied turbulent quantities (see [1] for a recent theoretical and experimental review). On the other hand, only recently some theoretical and experimental efforts have been done in order to understand the time properties of single-scale correlations, $C^{p,q}(r, r\,|t)$ [3,10] and the scaling properties of multi-scale correlations at zero-time delay, $C^{p,q}(r, R\,|0)$ [2,4–7].

In this paper, we propose and check a general phenomenological framework capable to capture all the above mentioned correlation functions and in agreement with the typical structure of non-linear terms of Navier–Stokes equations. In Section 2, the framework of the multifractal description of correlations is briefly recalled and critically examined. In Section 3, a representation for single-scale time correlations is introduced. In Section 4, we deal with the most generic two-scales time correlation. In Section 5, the phenomenological description proposed for fully developed Navier–Stokes turbulence is abridged for application to shell models of turbulence and its predictions are compared with the numerical results.

2. Background: the multifractal description of time correlations

One of the most important outcomes of experimental and theoretical analysis of turbulent flows is the spectacular ability of simple multifractal phenomenology [1,9] to capture the leading behavior of structure functions and of multi-scale correlation functions at zero-time delays [4,2,6]. This may appear not surprising because, as far as time-delays are not concerned, one may expect that (many) different phenomenological descriptions may well reproduce scaling laws typical of single-time correlation functions: multifractals being just one of these descriptions. More striking were the recent findings [3] that multifractal phenomenology may easily be extended to the time-domain such as to give a precise prediction on the behavior of the time properties of single-scale correlations. As soon as time enters in the game, one must ask consistency with the equation of motion: the major break-through was that one may write a time-multifractal description in agreement with the dynamics. We recall once more that when we refer to time-properties of turbulent flows we always mean the time-properties of the velocity fields once the trivial sweeping effects of large-scale on small-scales is removed.

Let us now quickly enter into the details of previous findings in order to clarify both the phenomenological framework and the notation that we will use in the following.

We remind that the multifractal (Parisi–Frisch) [9] description of single-time correlation functions is based on the assumptions that inertial range statistics is fully determined by a cascade process conditioned to some large-scale configuration:

$$\delta v_r = W(r, R) \cdot \delta v_R, \tag{2}$$

where the fluctuating function $W(r, R)$ can be expressed in terms of a superposition of local scaling solution $W(r, R) \sim (r/R)^{h(x)}$ with a scaling exponent $h(x)$ which assumes different values of h in a class of interwoven fractal sets with fractal co-dimension $Z(h) = 3 - D(h)$. From this assumption, one can write the expression for any structure functions of order m, which in our notation ($m = p + q$) becomes

$$S^m(R) \equiv C^{p,q}(R, R|0) \sim \langle W(R, L_0)^m \rangle \langle U_0^m \rangle, \tag{3}$$

$$S^m(R) \equiv \langle U_0^m \rangle \int \mathrm{d}\mu_{R,L_0}(h) \left(\frac{R}{L_0}\right)^{hm} \sim \left(\frac{R}{L_0}\right)^{\zeta(m)}, \tag{4}$$

where we have introduced the shorthand notation $\mathrm{d}\mu_{R,L_0}(h) \equiv \mathrm{d}h\, (R/L_0)^{Z(h)}$ to define the probability of having a local exponent h connecting fluctuations between scales R and L_0. Hereafter, as before, "$\sim$" means "equal within a scale independent constant". In (4) a steepest descent estimate was used, in the limit $R/L_0 \to 0$, in order to define the intermittent scaling exponents $\zeta(m) = \inf_h[Z(h) + mh]$. The signature of intermittency is the departure of the $\zeta(m)$ exponents from a linear behavior in m. In order to extend this description to the time-domain, it has been proposed [3] to consider that two velocity fluctuations, both at scale R but separated by a time-delay t, can be thought to be characterized by the same fragmentation process $W_{R,L_0}(t) \sim W_{R,L_0}(0)$ as long as the time separation t is smaller than the "instantaneous" eddy-turnover time of that scale, τ_R, while they must be almost uncorrelated for time larger than τ_R. Considering that the eddy-turnover time at scale R is itself a fluctuating quantity $\tau_R \sim R/(\delta v_R) \sim R^{1-h}$, we may write down [3]:

$$C^{p,q}(R, R|t) \sim \int \mathrm{d}\mu_{R,L_0}(h) \left(\frac{R}{L_0}\right)^{h(p+q)} F_{p,q}\left(\frac{t}{\tau_R}\right), \tag{5}$$

where the time-dependency is hidden in the function $F_{p,q}(x)$ which must be a smooth function of its argument (for example a decreasing exponential). From (5) it is straightforward to realize that at zero-time separation we recover the usual structure function representation. It is much more interesting to notice that (5) is also in agreement with the constraints imposed by the non-linear part of the Navier–Stokes equations. Indeed, to make short a long story (see [10] for a rigorous discussion) we may say that under the only hypothesis that non-linear terms are dominated by local interactions in the Fourier space we can safely assume that as far as power-law counting is concerned the inertial terms of Navier–Stokes equations for the velocity difference δv_R can be estimated to be of the form

$$\partial_t \delta v_R(t) \sim O\left[\frac{(\delta v_R(t))^2}{R}\right], \tag{6}$$

and, therefore we may check that

$$\partial_t C^{p,q}(R, R|t) \sim \int \mathrm{d}\mu_{R,L_0}(h) \left(\frac{R}{L_0}\right)^{h(p+q)} (\tau_R)^{-1} F'_{p,q}\left(\frac{t}{\tau_R}\right) \sim \frac{C^{p+1,q}(R, R|t)}{R}, \tag{7}$$

where of course in the last relation there is hidden the famous closure-problem of turbulence, now restated in term of the relation: $\frac{\mathrm{d}}{\mathrm{d}t} F_{p,q}(t) \sim F_{p+1,q}(t)$. Let us therefore stress that we are "not solving turbulence" but just building up a phenomenological framework where all the leading (and subleading, see below) scaling properties are consistent with the constraints imposed by the equation of motion. [1]

[1] In order to really attack the NS equations in this framework one should dive into the structure of the F_{pq}-functions in great detail: a problem which seems still to be far from convergence [10].

In the following we shall show how the representation (5) must be improved to encompass the most general multi-time, multi-scale correlation $C^{p,q}(r, R|t)$.

3. Single-scale time correlations

We shall first show in which respect expression (5) may not be considered a satisfactory representation of single-scale time correlation. The first comment that can be raised about (5) is that it misses important subleading terms which may completely spoil the long-time scaling behavior: indeed, the main hypothesis that correlation $C^{p,q}(R, R|t)$ feels only the eddy-turnover time of the scale R is too strong. It is actually correct only when the correlation function has zero disconnected part, i.e. when $\lim_{t\to\infty}\langle\delta v_R^p(0)\delta v_R^q(t)\rangle \equiv 0$ which is certainly false in the most general case. The problem is not only limited to the necessity of taking into account the asymptotic mismatch to zero given by the disconnected terms, $\langle\delta v_R^p\rangle\langle\delta v_R^q\rangle$, – which would be a trivial modification of (5) – because as soon as the disconnected part is present the whole hierarchy of fluctuating eddy-turnover times from the shortest, τ_R, up to the largest, τ_{L_0}, must be felt by the correlation.

Let us, for the sake of simplicity, introduce a hierarchical set of scales, $l_n = 2^{-n}L_0$ with $n = 0, \ldots, n_d$, which span the whole inertial range and let us simplify the notation by taking $L_0 = 1$ and by writing $u_n = \delta v_r$ in order to refer to a velocity fluctuation at scale $r = l_n$.

More precisely, we can perform a wavelet decomposition of the field of velocity differences in a Quasi-Lagrangian frame of reference: then u_n stands for a representative of the wavelet coefficients at the octave n.

The picture which will allow us to generalize the time-multifractal representation to the multi-time, multi-scale case goes as follows.

For time-delays, $t \sim \tau_m$, typical of the eddy-turnover time of the mth scale we may safely say that the two velocity fluctuations follow the same fragmentation process from the integral scale L_0 down to scale l_m while they follow two uncorrelated processes from scale l_m down to the smallest scale in the process, l_n. In the multifractal language we must write that for time $t \sim \tau_m$ we have

$$u_n(0) \sim W'_{n,m}(0)W_{m,0}(0)u_0(0) \sim \left(\frac{l_n}{l_m}\right)^{h'}\left(\frac{l_m}{L_0}\right)^{h} u_0(0), \tag{8}$$

$$u_n(t) \sim W''_{n,m}(t)W_{m,0}(t)u_0(t) \sim W''_{n,m}(t)W_{m,0}(0)u_0(0) \sim \left(\frac{l_n}{l_m}\right)^{h''}\left(\frac{l_m}{L_0}\right)^{h} u_0(0), \tag{9}$$

where with $W, W', W'', \ldots$ we mean different independent outcomes of the cascade process with exponents h, h', h'' and where we have used the fact that in this time-window $W_{m,0}(t) \sim W_{m,0}(0)$.

Apart from subtle further-time dependencies (see below) we should therefore conclude that for time $t \sim \tau_m$ the correlation functions may be approximated as

$$C^{p,q}_{n,n}(\tau_m) \sim \langle W^p_{n,m}\rangle\langle W^q_{n,m}\rangle\langle W^{p+q}_{m,0}\rangle, \tag{10}$$

which must be considered the fusion-rules prediction for the time-dependent fragmentation process [4,2,6]. Let us notice that this proposal has already been presented in [8] and considered to express the leading term in the limit of large time-delays $\tau_m \to \infty$; here, we want to refine the proposal made in [8] showing that by adding the proper time-dependencies it is possible to obtain a coherent description of the correlation functions for all time-delays. Expression (10) summarizes the idea that for time-delay larger than τ_m but smaller than τ_{m-1}, velocity components with support on scales $r > l_{m-1}$ did not have enough time to relax and therefore the local exponent, h, which describes fluctuations on those scales must be the same for both fields. On the other hand, components with support on scales $r < l_{m-1}$ have already decorrelated for $t > \tau_{m-1}$ and therefore we must consider two independent scaling exponents h', h'' for describing fluctuations on these scales.

Adding up all these fluctuations, centered at different time-delays, we end with the following multifractal representation for $C^{p,q}(l_n, l_n|t) \equiv C^{p,q}_{n,n}(t)$:

$$C^{p,q}_{n,n}(t) = \int \mathrm{d}\mu_{n,0}(h) l_n^{(q+p)h} F_{p,q}\left(\frac{t}{\tau_n}\right) + \sum_{m=1}^{n-1} \int \mathrm{d}\mu_{m,0}(h)\mathrm{d}\mu_{n,m}(h_1)\mathrm{d}\mu_{n,m}(h_2)\left(\frac{l_m}{L_0}\right)^{(q+p)h}\left(\frac{l_n}{l_m}\right)^{qh_1}\left(\frac{l_n}{l_m}\right)^{ph_2} f_{p,q}\left(\frac{t}{\tau_m}\right). \tag{11}$$

Let us now spend a few words in order to motivate the previous expression. In the first term of the R.H.S. of (11) we have explicitly separated the only contribution we would have in the case of vanishing disconnected part. This term remains the leading contribution in the static limit ($t = 0$) also when disconnected parts are non-zero. About the new terms controlling the behavior of the correlation functions for larger time the most general dependency should include in the arguments of $f_{p,q}$ also the ratios $(l_n/l_m)^{1-h_1}$ and $(l_n/l_m)^{1-h_2}$. In first approximation, we assume the simplified dependence in (11) with $f_{p,q}(x)$ being a function peaked at its argument $x \sim 0(1)$ which must be exactly zero for $x = 0$ and different from zero only in a interval of width $\delta x \sim O(1)$.

Let us now face the consistency of the representation (11), with the constraint imposed by the equations of motion. By applying a time-derivative to a correlation $C^{p,q}_{n,n}(t)$ one produces a new correlation with by-definition zero-disconnected part, whose representation has thus no subleading term ($f_{p,q} \equiv 0$). When performing the time-derivative on both sides of (11) it is evident that – in order to accomplish the dynamical constraints – all the derivatives of the subleading terms must sum to a zero contribution.

This is the first non-trivial result we have reached until now. If our representation (11) is correct, we claim that all eddy-turnover times must be present in the general single-scale correlation functions but strong cancellations of all subleading terms must take place whenever disconnected contributions vanish.

Let us finally notice that for time-delays larger than the eddy-turnover time of the integral scale we should add to the RHS of (11) the final exponential decay toward the full disconnected term $\langle u_n^p\rangle\langle u_n^q\rangle$.

4. Two-scales time correlations

Let us now jump to the most general multi-scale, multi-time correlation functions:

$$C^{p,q}(r, R|t) = \langle \delta v_R^q(0) \cdot \delta v_r^p(t)\rangle, \tag{12}$$

where from now on we will always suppose that δv_r describes the velocity fluctuation at the smallest of the two scales considered, i.e. $r < R$. It is clear that now we have to consider the joint statistics of two fields: first, the slower, at large-scale, $\delta v_R(0)$ and second, the faster, at small-scale and at a time-delay t, $\delta v_r(t)$.

As done in the previous section, we shall use an octave-based notation $u_n = \delta v_R$ and $u_N = \delta v_r$, where it is understood that $r = 2^{-N} L_0$ and $R = 2^{-n} L_0$ (with $N > n$), denoting $C^{p,q}(r, R|t) \equiv C^{p,q}_{N,n}(t)$.

Following the same reason as before we may safely assume that from zero time-delays up to time delays of the order of the slower component, $t \le \tau_n$, the velocity field at small-scale feels the same transfer process of u_n up to scale n and then from scale n to scale N an uncorrelated transfer mechanism:

$$u_N(t) = W_{N,n}(t) u_n(t) \sim W_{N,n}(t) u_n(0) \sim W_{N,n}(t) W'_{n,0}(0) u_0 \quad \text{for} \quad 0 \le t \le \tau_n. \tag{13}$$

Similarly, for time-delays within $\tau_n \le \tau_m < t < \tau_{m-1} \le \tau_0$ also the field at large-scale n will start to see different transfer processes:

$$u_n(0) \sim W''_{n,m}(0) u_m(0) \sim W''_{n,m}(0) W'_{m,0}(0) u_0, \tag{14}$$

$$u_N(t) \sim W_{N,m}(t) u_m(t) \sim W_{N,m} u_m(0) \sim W_{N,m}(t) W'_{m,0}(0) u_0. \tag{15}$$

It is clear now, how we may write down the correlation for any time:

$$C^{p,q}_{N,n}(t) = \int \mathrm{d}\mu_{n,0}(h)\mathrm{d}\mu_{N,n}(h_1) \left(\frac{l_n}{L_0}\right)^{(q+p)h} \left(\frac{l_N}{l_n}\right)^{ph_1} F_{p,q}\left(\frac{t-\tau_{nN}}{\tau_n}\right) + \sum_{m=1}^{n-1} \int \mathrm{d}\mu_{m,0}(h)\mathrm{d}\mu_{n,m}(h_1)\mathrm{d}\mu_{N,m}(h_2) \left(\frac{l_m}{L_0}\right)^{(q+p)h} \left(\frac{l_n}{l_m}\right)^{qh_1} \left(\frac{l_N}{l_m}\right)^{ph_2} f_{p,q}\left(\frac{t}{\tau_m}\right) \tag{16}$$

where $\tau_{nN} \simeq \tau_n - \tau_N$ represents the time-delay needed for an energy burst to travel from shell n to shell N and the functions $F_{p,q}(x)$ and $f_{p,q}(x)$ are defined in the same way as done for (11).

Let us observe that the sum in the above expression goes only up to the index of the largest scale n: this is because only for time larger than τ_n the correlation is a true multi-time correlation. Indeed, for time-delay shorter than τ_n only the field at small-scale, u_N, is changing but always under the same large-scale configuration, u_n.

Let us stress that in general $F_{p,q}$ and $f_{p,q}$ should depend on all the time-scale ratios into the game: $(l_n/l_m)^{1-h_1}$, $(l_N/l_m)^{1-h_2}$. We have here assumed that in first approximation the main effect of the asymmetry in n, N is a delay in the propagation of the correlation. Other, more complex functional dependences are possible but we think that it would be very difficult to discriminate between them.

The matching of representation, (16), with the equation of motion reveals some important dynamical properties. From simple time-differentiation we should have

$$\partial_t C^{p,q}_{N,n}(t) \sim O\left[\frac{C^{p+1,q}_{N,n}(t)}{l_N}\right], \tag{17}$$

which seems to be in disagreement with the time-representation proposed (16) because in the RHS of (17) does appear explicitly the fast eddy-turnover time τ_N (through the dependency from l_N). Actually, the representation (16), is still in agreement with the equation of motion because the dependency of (17) from τ_N is false: again, exact cancellations must take place in the RHS. The explanation goes as follows: in the multifractal language we may write $u_N(t) = W_{N,n}(t)u_n(t)$, and therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}u_N(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}W_{N,n}(t)\right)u_n(t) + W_{N,n}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}u_n(t)\right), \tag{18}$$

but for time shorter than the eddy-turnover of the large-scale τ_n, the term $W_{N,n}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u_n(t)$ is zero because the shell u_n did not move at all, while, once averaged, the first term of the RHS of (18) becomes $\left\langle\left(\frac{\mathrm{d}}{\mathrm{d}t}W_{N,n}(t)\right)\right\rangle\langle u_n(t)\rangle$ which also vanishes because of the total time-derivative. The time-derivative, $\partial_t C^{p,q}_{n,N}(t)$ will therefore be a function which scales as $C^{p,q+1}_{N,n}(t)/l_n$ instead of $C^{p+1,q}_{N,n}(t)/l_N$ as simple power counting would predict. This may even be shown rigorously by evaluating the following averages:

$$\partial_t\langle u_n^q(t)u_N^p(t)\rangle \equiv 0 \equiv \langle(\partial_t u_n^q)u_N^p\rangle + \langle u_n^q(\partial_t u_N^p)\rangle. \tag{19}$$

The previous exact relation forces one of the two correlation to not satisfy the simple multifractal ansatz because otherwise power-law counting would be contradictory:

$$\langle(\partial_t u_n^q)u_N^p\rangle \sim \frac{C^{p,q+1}_{N,n}(0)}{l_n} \neq \langle u_n^q(\partial_t u_N^p)\rangle \sim \frac{C^{p+1,q}_{N,n}(0)}{l_N}. \tag{20}$$

Now, in view of the previous discussion, we know that it is the correlation with the time-derivative at small-scale, $\langle u_n^q(\partial_t u_N^p)\rangle$, which does not satisfy the multifractal power-law, but has the same scaling of $\langle(\partial_t u_n^q)u_N^p\rangle$ as our representation correctly reproduces.

5. Shell models

In order to check the representations (11) and (16) we have performed numerical investigations in a class of dynamical models of turbulence (shell models) [11]. Within this modeling the approximation of local-interactions among velocity fluctuations at different scales is exact and therefore no-sweeping effects are present. This fact makes of shell models the ideal framework where non-trivial temporal properties can be investigated.

We shall adopt the following dynamical equations for the complex shell variables u_n (further details on these model equations are given in Ref. [12])

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = ik_n(u_{n+2}u_{n+1}^* - \frac{1}{4}u_{n+1}u_{n-1}^* + \frac{1}{8}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n. \tag{21}$$

Shell variables u_n are meant to represent velocity fluctuations δv_R at scale $R = 2^{-n}L_0 = k_n^{-1}$. The total number of shells in our simulations is 24. Forcing is restricted to the first and second shell and the viscosity coefficient is $\nu = 5 \times 10^{-7}$ corresponding to a Reynolds number $Re \simeq 10^8$. A statistically steady state is reached after a transient of a few large eddy-turnover times, and then time-averages are computed over several eddy-turnover times.

5.1. Single-scale time correlations

In order to test the dependency of (11) from the whole set of eddy-turnover times we show in Fig. 1 the correlation $C_{nn}^{p,q}(t)$ for two cases with and without disconnected part. As it is evident, the correlation with a non-zero disconnected part decays in a time-interval much longer than the characteristic time of the shell τ_n. This shows that it is not possible to associate a single time-scale τ_n to the correlation functions of the form (11). In Fig. 2, we also compare the correlation when one of the two observable is a time-derivative with the correlation chosen such as to have the same dimensional properties but without being an exact time-derivative. Also in this case the difference is completely due to the absence (presence) of all subleading terms in the former (latter).

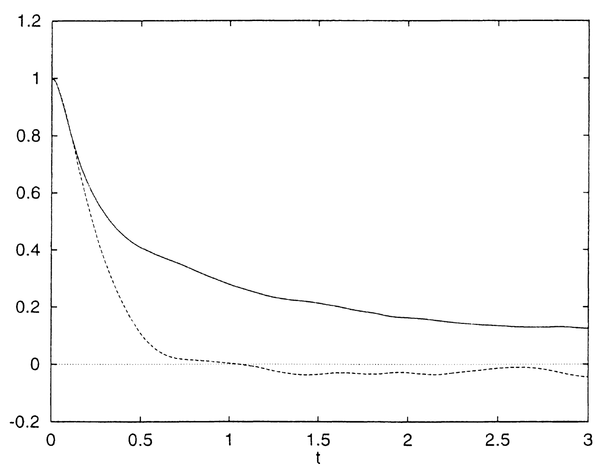

Fig. 1. The time-dependency of single-scale correlation functions, $C_{n,n}^{1,1}(t)$, in two different cases. The continuous line is the case with a non-zero disconnected part, $\langle |u_n(0)\|u_n(t)|\rangle - \langle |u_n|\rangle^2$, while the dashed line represents a case with vanishing disconnected part $\Re(\langle u_n(0)u_n^*(t)\rangle)$. Both correlations are rescaled to their value at $t = 0$. The scale is fixed in the middle of the inertial range, $n = 12$, and the eddy-turnover time of the reference scale was $\tau_{12} \simeq 0.29$. The average has been performed over approximately $500\tau_{12}$, about 10 large eddy-turnover times. The presence of subleading terms in the first case (continuous line) is apparent. The remnant anticorrelation in the second case, for $t > \tau_{12}$, reveals a partial cancellation of subleading terms: full cancellation requires averaging over a time interval of many more large eddy-turnover times.

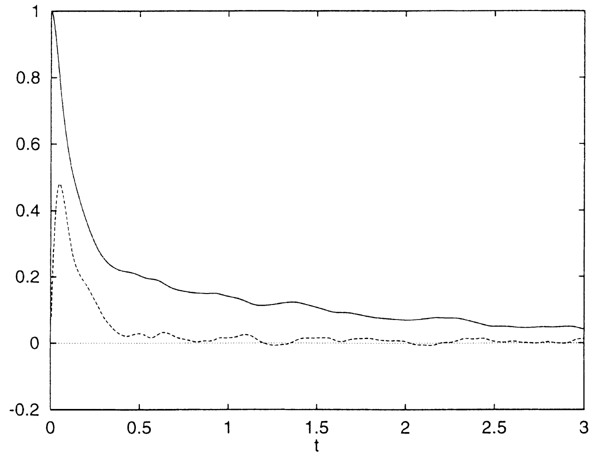

Fig. 2. Comparison between the two correlations $k_n\langle|u_n(0)|^2\,|u_n(t)|^3\rangle$, continuous line. $-\langle|u_n(0)|^2\,\mathrm{d}|u_n(t)|^2/\mathrm{d}t\rangle$, dashed line. The two correlations have the same dimensional properties, but the latter decays faster due to cancellations of subleading terms. It vanishes at zero delay because of stationarity and smoothness of the process $u_n(t)$. Scale and characteristic times as in Fig 1.

5.2. Intermittent integral time-scales

In the case when the disconnected part of the time correlation is absent, in the representation (11) all the subleading terms mutually cancel, letting the fully-connected contribution alone.

Under this condition and in presence of intermittency one expects anomalous scaling behavior for the integral time-scales, $s^{(p,q)}(R)$, characterizing the mean decorrelation time of fluctuations at scale R, defined as [3]:

$$s^{(p,q)}(R) = \frac{\int_0^\infty \mathrm{d}t\, C^{p,q}(R,R|t)}{C^{p,q}(R,R|0)}, \tag{22}$$

exploiting the multifractal representation (11) it is easy to show that

$$s^{(p,q)}(R) = s^{(p+q)}(R) \sim \left(\frac{R}{L_0}\right)^{z(p+q)}, \tag{23}$$

where the exponents $z(m)$ are fully determined in terms of the intermittent spatial scaling exponents: $z(m) = 1 + \zeta(m-1) - \zeta(m)$.

This prediction in practice is very difficult to check: indeed full cancellation of the subleading terms requires an extremely long time span, and since the cancellations affect dramatically the convergence of the time integral there is no chance of measuring with sufficient precision the $z(m)$ exponents.

In order to bypass this problem we devised an alternative way to extract the integral times. We introduce fluctuating decorrelation times at a scale R, defined as the time interval T_i in which the instantaneous value of the correlation has changed by a fixed factor λ, i.e. in our octave notation:

$$u_n(t_i)u_n(t_i+T_i) = \lambda^{\pm 1}|u_n(t_i)|^2. \tag{24}$$

At time $t_{i+1} = t_i + T_i$ we repeat this procedure and we record the new decorrelation time T_{i+1} and so forth for an overall number of trials $\mathcal{N}$. The averaged decorrelation times can be thus defined as

$$\tau_n^{(m)} = \langle T^2|u_n|^m\rangle_e / \langle T|u_n|^m\rangle_e = \langle T|u_n|^m\rangle_t / \langle |u_n|^m\rangle_t,$$

Table 1
Comparison between the integral time-scales intermittency exponents, z_m, estimated from the measured spatial intermittent exponents, $z_m^{(\mathrm{th})} = 1 + \zeta(m-1) - \zeta(m)$, and from direct measuring via the "doubling-time" T, $z_m^{(\mathrm{num})}$

m	ζ_m	$z_m^{(\mathrm{num})}(z_m^{(\mathrm{th})})$
1	0.39	0.61(0.61)
2	0.72	0.68(0.67)
3	1.00	0.72(0.72)
4	1.26	0.75(0.74)
5	1.49	0.77(0.78)
6	1.71	0.78(0.78)
7	1.93	0.80(0.80)
8	2.13	0.80(0.80)

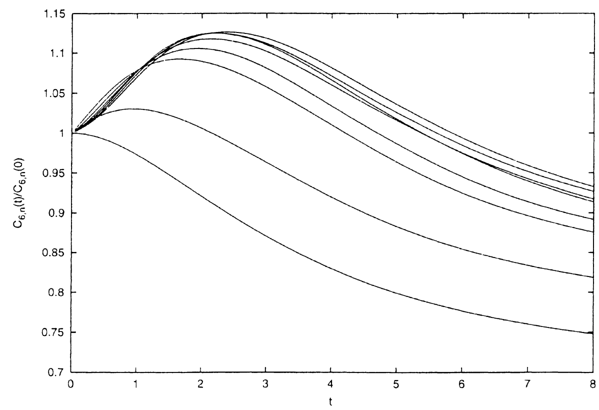

Fig. 3. Multi-time multi-scale correlation functions, $C^{1,1}_{n,N}(t)$, for $n = 6$ and $N = 6, \ldots, 13$ (from bottom curve to top curve). Observe the saturation in the time-delays, $\tau_{n,N} \equiv \tau_n - \tau_N \to \tau_n$ when N increases.

where $\langle \cdots \rangle_e$ stands for ensemble averaging over the $\mathcal{N}$ trials and $\langle \cdots \rangle_t$ represents the usual time average,[2] . Since the multifractal description applies to time averages the averaged decorrelation times scale as $\tau_n^{(m)} \sim l_n^{z(m)}$ with the same scaling exponents of the integral times $s_n^{(m)}$.

In Table 1, we report the observed numerical $\zeta(m)$ along with the observed and expected scaling exponents for $\tau_n^{(m)}$, showing a very good agreement.

5.3. Two-scales time correlations

In order to test the representation (16) we plot in Fig. 3 the typical multi-time, multi-scale velocity correlation $C^{p,q}_{N,n}(t)$ for $p = q = 1$, $n = 6$, $N = 6 - 13$. As one can see the correlation has a peak which is in agreement with the delay predicted by (16), which saturates at the value $\tau_{nN} \simeq \tau_n$ for $n \ll N$.

Let us also notice that due to the dynamical delay, $\tau_{m,N}$, the simultaneous multi-scale correlation functions $C^{p,q}_{N,n}(0)$ do not show the fusion-rules prediction, i.e. pure power laws behaviors at all scales:

$$C^{p,q}_{N,n}(0) \sim \frac{S_N^p}{S_n^p} S_n^{p+q} \sim \left(\frac{l_N}{l_n}\right)^{\zeta(p)} \left(\frac{l_n}{L_0}\right)^{\zeta(p+q)}, \tag{25}$$

[2] The relation between the e-average and the t-average is simply derived by observing that $\langle |u_n|^m T\rangle_t = (\int_0^{\mathcal{T}} T|u_n|^m \mathrm{d}t)/\mathcal{T} \simeq (\sum_i T_i^2 |u_n(t_i)|^m)/(\sum_i T_i) = \langle T^2|u_n|^m\rangle_e / \langle T\rangle_e$.

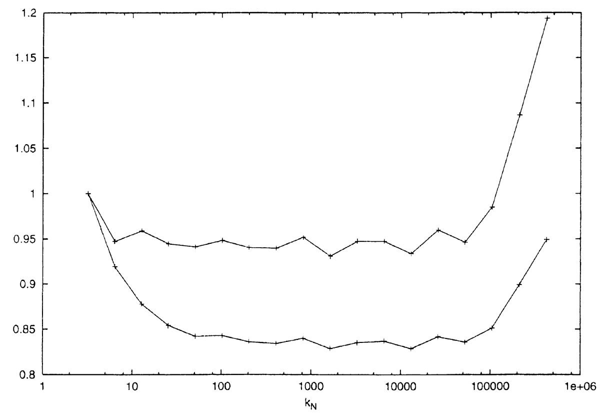

Fig. 4. Lin-log plot of Multi-scale correlation $C^{1,1}_{n,N}(t) = \langle |u_n(0)|\,|u_N(t)|\rangle$ rescaled with the Fusion Rule prediction: $C^{1,1}_{n,N}(t)/(S^1_N S^2_n/S^1_n)$ at fixed $n = 6$ and at changing $N \geq n$. The lower line represents the zero-delay correlation ($t = 0$), the upper line is for the average delay $t = T_{6,N}$.

where $S^p_n \equiv \langle |u_n|^p\rangle$ is the pth order structure function. As a matter of fact for $t \to 0$, only the first term on the RHS survives in (16) and $F_{p,q}(-\tau_{nN}/\tau_n)$ can be considered a constant only in the limit of large scale separation, $n \ll N$, while otherwise we will see finite-size corrections.

The effect of the delay in multi-scale correlations is shown in Fig. 4, where we compare $C^{1,1}_{N,n}(0)$ and $C^{1,1}_{N,n}(T_{nN})$ (for $N > n = 6$) rescaled with the Fusion Rule prediction (25). The time-delay T_{nN} is the time of the maximum of $C^{1,1}_{N,n}(t)$ computed from Fig. 3. We see that without delay, the prediction (25) is recovered only for $N \gg n$ with a scaling factor $F_{1,1}(-1) \simeq 0.83$, while including the average delay T_{nN} the Fusion Rule prediction is almost verified over all the inertial range.

Of course the delay τ_{nN} is a fluctuating quantity and one should compute the average (16) with fluctuating delays. In this case the dimensional estimate $\tau_{nN} = l_n u_n^{-1} - l_N u_N^{-1}$ is somehow ill-defined, first of all being not positive definite. To find a correct definition for the fluctuating time-delays is a subtle point which lays beyond the scope of the present Paper.

6. Conclusions

In conclusion, we have proposed a multifractal-like representation for the multi-time, multi-scale velocity correlation which should take into account all possible subtle time-dependencies and scale-dependencies. The proposal can be seen as a merging of the proposal made in [3] – valid only for cases when the disconnected part is vanishing – and the proposal made in [8] – valid only in the asymptotic regime of large time-delays and large-scale separation. Our proposal is phenomenologically realistic and consistent with the dynamical constraints imposed by the equation of motion. We have numerically tested our proposal within the framework of shell models for turbulence. A new way to measure intermittent integral-time scales, $s^{(p,q)}(R)$, has also been proposed and tested.

Further tests on the true Navier–Stokes equations would be of first-order importance. Furthermore, the building of synthetic signals which would reproduce the correct dynamical properties of the energy cascade would also be of primary importance [13].

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