Physica A 185 (1992) 19–27
North-Holland

PHYSICA A

Multifractality in a shell model for 3D turbulence

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A shell model with few degrees of freedom is analyzed to test the multifractal prediction for the intermittent behaviour in turbulent flows. It takes into account the main phenomenological aspects of turbulence in 3D: cascade of energy toward small scales and local interactions in the wave numbers.

Although it is not realistic, it permits an investigation of the dynamical mechanisms leading to non-trivial scaling laws and multifractality in turbulence. We find that the energy dissipation in the model has indeed a fractal structure, which can be explained in terms of the time intermittency in the chaoticity degree, by computing the effective maximum Lyapunov exponent and eigenvector. We also compute the probability distribution of the velocity gradients and the scaling laws of the structure functions, which are found to be in good agreement with the experimental and numerical results.

A theoretical framework for an analytical calculation of the multifractal spectra is described in the conclusion.

1. Introduction

The small scale statistics of three-dimensional fully developed turbulence is one of the fundamental problems in fluid mechanics. The phenomenological theory of Kolmogorov [1] gives a qualitatively correct description of the main mechanisms acting in incompressible fluids at high Reynolds number Re. In turbulent flows, there is a cascade transfer of energy toward the small scales where the dissipation, due to molecular friction, plays the fundamental role. The cascade is hierarchical in the sense that a disturbance on a certain scale receives its energy from a larger scale disturbance and transfer it to smaller scale disturbances. At the end of the cascade, one has the direct conversion of the smallest disturbances into heat. Moreover, the velocity gradients are very large.

Assuming a constant rate of non-linear transfer of energy one obtains the classical Kolmogorov results. Dimensional analysis suggests that the Navier–Stokes equations have singular velocity gradients in the limit of infinite

Reynolds number, i.e., the velocity difference $\delta v(l) \equiv |\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{l}) - \boldsymbol{v}(\boldsymbol{x})| \sim l^h$ where $l = |\boldsymbol{l}|$, with a singularity $h = 1/3$. It follows that, in the inertial range, the velocity structure functions scale as

$$\langle \delta v(l)^Q \rangle \propto l^{\zeta_Q}, \qquad \text{with } \zeta_Q = Q/3\,, \tag{1}$$

where $\langle \cdot \rangle$ is a spatial average.

Nevertheless, there are many experimental [2] and numerical [3] evidences that strong fluctuations of the energy transfer and dissipation are present, leading to the existence of a whole spectrum of possible singularities. In particular, the exponents ζ_Q are different from their classical value $Q/3$ and appear to be non-linear in Q. Some fractal phenomenological approaches have been proposed in the last years [4] to explain the intermittent behavior. We believe that a first goal is the connection between the corrections to the Kolmogorov scaling and the dynamical properties of the time evolution generated by the Navier–Stokes equations.

For this reason, it is useful to analyze particular models of the energy cascade process, instead of the complete Navier–Stokes equations, using an approach to the intermittency problem firstly proposed by Obukhov [5], Gledzer [6], Siggia [7] and developed by Grappin et al. [8].

2. A shell model

We study a shell model [9] where the Fourier space is divided in N shells.

Each shell k_n ($n = 1, 2, \ldots, N$) consists of the wave numbers k such that $K_0 2^n < k \leq K_0 2^{n+1}$. The velocity difference over a length scale $\approx k_n^{-1}$ is given by u_n. The energy is $E = \Sigma\, |u_n|^2/2$ and its power spectrum is $E(k_n) = \langle |u_n|^2 \rangle / 2k_n$. The Navier–Stokes equations are thus approximated by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_n^2\right) u_n = \mathrm{i}(a_n u_{n+1}^* u_{n+2}^* + b_n u_{n-1}^* u_{n+1}^* + c_n u_{n-1}^* u_{n-2}^*) + f\delta_{n,4}\,, \tag{2}$$

where ν is the viscosity, and f is a forcing (here on the fourth mode).

There are two main qualitative differences with Navier–Stokes:

(i) k is a scalar (no spatial structures).

(ii) There are only nearest neighbor interactions among shells. From demanding energy conservation when $\nu = f = 0$, one has

$$\begin{aligned} &a_n = k_n\,, \qquad b_n = -\tfrac{1}{2}k_{n-1}\,, \qquad c_n = -\tfrac{1}{2}k_{n-2}\,, \\ &b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0\,. \end{aligned} \tag{3}$$

The unstable fixed point of eqs. (2) when $\nu = f = 0$ is given by the Kolmogorov scaling $u_n \propto k_n^{-1/3}$. The time evolution given by (2) exhibits a chaotic behaviour on a strange attractor in the $2N$-dimensional phase space, with a maximum Lyapunov exponent proportional to $\nu^{-1/2}$ [8, 9].

The numerical integrations of eq. (2) have been performed using two different algorithms: fourth-order Runge–Kutta and Burlirsch–Stoer, with 16 digits precision. We have considered $N = 19$ shells with $\nu = 10^{-6}$, $f = (1 + \mathrm{i}) \times 5 \times 10^{-3}$, $K_0 = 2^{-4}$, and $N = 27$ with $\nu = 10^{-9}$, $f = (1 + \mathrm{i}) \times 5 \times 10^{-3}$, $K_0 = 0.05$.

3. The multifractal interpretation

The energy spectrum $E(k)$ is observed to scale as $k^{-\alpha}$, in the inertial range, with an exponent $\alpha = 1 + \zeta_2 \sim 1.7$ not exactly equal to the value 5/3 expected by applying dimensional arguments. In fig. 1 one sees that in the model, the exponents ζ_Q are not linear in Q, and can be fitted by the random β model [10] formula

$$\zeta_Q = Q/3 - \ln_2[1 - x + x(\tfrac{1}{2})^{1-Q/3}]\,, \qquad x = 0.12\,, \tag{4}$$

where only two possible kinds of fragmentation are assumed in the cascade process: a disturbance generates either vorticity sheets (with probability x) or space filling disturbances, as in the Kolmogorov theory (with probability $1 - x$).

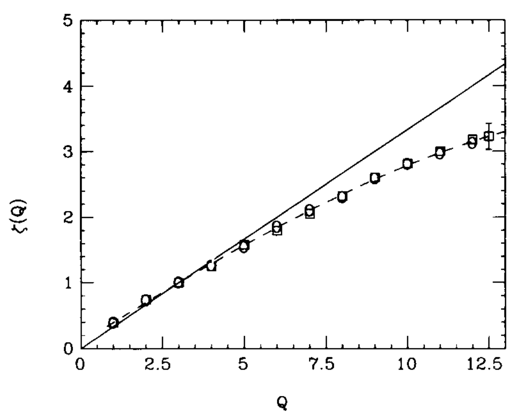

Fig. 1. The structure function exponent ζ_Q, plotted vs Q for positive integer Q up to 12. The squares are obtained by an integration of eq. (2) with $N = 27$ shells, the circles with $N = 19$. The errors are smaller than the size of the symbols for $Q < 10$ and for $Q \geqslant 10$ are given by the bar close to the last point. The solid line is the Kolmogorov result $\zeta_Q = Q/3$; the dashed line is the random β model fit of eq. (4).

The value for the only free parameter x is very near to that used to fit the experimental data of Anselmet et al. [2] ($x = 0.125$).

The intermittency of the energy dissipation exhibit by the model is therefore consistent with the multifractal approach [10], where one considers a hierarchy of singularities h and related fractal sets S(h) of fluid points $\boldsymbol{x}$, such that $|\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{l}) - \boldsymbol{v}(\boldsymbol{x})| \sim l^h$. The fractal dimensions $D(h)$ of these sets are related to the exponents ζ_Q by the Legendre transformation [10]

$$\zeta_Q = \min_h \,[hQ - D(h) + 3]\,. \tag{5}$$

4. Temporal intermittency

We want to link the multifractal corrections with the behavior of the instantaneous maximum Lyapunov exponent and of its eigenvector. The spectrum of Lyapunov exponents $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{2N}$ can be computed [11] by considering the linear variational equation

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = A_{i,j} z_j\,, \qquad i, j = 1, \ldots, 2N\,, \tag{6}$$

for the time evolution of an infinitesimal increment $z = \delta U$, where $A_{n,j} \equiv \partial F_n / \partial U_j$ is the Jacobian matrix of eqs. (2), and $U \equiv (u_1, u_2, \ldots, u_N)$.

The solution for the tangent vector z can thus be formally written as $z(t_2) = M(t_1, t_2)\, z(t_1)$, with $M = \exp \int_{t_1}^{t_2} A(\tau)\,\mathrm{d}\tau$. The orthonormal Lyapunov basis is then given by the $2N$ eigenvectors f_i of the matrix $M^\dagger M$ in the limit $t \to \infty$, and depends on the starting point U_0 in the phase space. It is also possible to introduce [12] a stability basis $\boldsymbol{e}_i$ given by the eigenvectors of the matrix M. Note that a generic tangent vector $z(t)$ is projected by the evolution along $\boldsymbol{e}_1$, i.e. $z(t) = c \exp(\lambda_1 t)\, \boldsymbol{e}_1$, a part corrections $\mathcal{O}(\exp - |\lambda_1 - \lambda_2| t)$. Moreover there is a strong correspondence between Lyapunov eigenvectors of the last negative Lyapunov exponents and dissipative modes following the end of the inertial scaling range. This result is somewhat expected since the viscous damping is responsible of the strongest contraction rates, so that $\lambda_{2i} \propto -\nu k_i^2$ for $i \approx N$.

More interesting, a large part of the Lyapunov exponents is found to be very close to zero. Their eigenvectors are directed along directions in phase space given by the inertial wave number shells. This has suggested [9, 13] that power scaling laws in turbulence are connected to the large number of marginal eigenvalues in the spectrum of $M^\dagger M$.

However, the maximum Lyapunov exponent is proportional to the inverse of the smallest characteristic time of the system, the Kolmogorov turnover time. It is expected to be the origin of the intermittent corrections to power laws connected with the almost zero Lyapunov exponents. In fact, the average of the first eigenvector $\boldsymbol{e}_1$ is not concentrated on the small wave numbers, but spreads in the whole inertial range [9].

We have computed the response after a time τ to an infinitesimal perturbation, defining an instantaneous maximum Lyapunov exponent as

$$\chi_\tau(t) \equiv \frac{1}{\tau} \ln \left| \frac{z(t+\tau)}{z(t)} \right| . \tag{7}$$

The value of χ is an indication of the global chaoticity of the system, at a given instant. The squared modulus of the projection of the first eigenvector on the nth shell $p(n) \equiv |e_1(k_n)|^2/\Sigma_j |e_1(k_j)|^2$ can be interpreted as the fraction of the largest instability localized over the shell k_n.

In the laminar phase the values of $p(n)$ for different n spread around the forced mode and over the whole inertial range while in the chaotic regime they are significantly different from zero only for a few shells k_n around a dissipative shell k_{D}. This suggests that a solution of eq. (2) spends most of the time around the Kolmogorov fixed point. The intermittent behavior is thus produced by strong bursts of chaoticity along a direction corresponding to the dissipative shells followed by a contraction back to the fixed point.

In order to give a quantitative description of the above scenario, we have focused the attention on one dissipation wave number $k_{15} = k_{\mathrm{D}}$ in a numerical integration with $N = 19$ shells. The instantaneous Lyapunov exponent χ, the energy dissipation E_{D} estimated by $|u_{15}|^2$, and the chaoticity fraction on the shell k_{D}, estimated by $p_{\mathrm{D}} \equiv p(k_{15})$, present a very strong temporal intermittency. The peaks of the three temporal sequences are moreover very correlated. This indicates that instantaneously the chaoticity concentrates on dissipative wave numbers, in correspondence with high values of the energy dissipation and of the instantaneous Lyapunov exponent [9]. We have computed the correlations between E_{D}–p_{D} and E_{D}–χ; they show a very fast decaying in time. There is a strong anticorrelation between E_{D} and χ after a delay of order one time unit. This provides evidence that a chaoticity burst is followed from a strong contraction rate (i.e. a negative χ).

5. The probability distribution function (PDF) of the velocity gradients

One of the fundamental features of the three-dimensional fully developed turbulence is the non-Gaussian statistics of the velocity gradients. Here we present our result obtained applying the multifractal picture [14]. In this case the velocity increments $\delta v_x(l) = |\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{l}) - \boldsymbol{v}(\boldsymbol{x})|$ are assumed to scale as $\delta v_x(l) \propto v_0 l^h$, where $v_0 = |\boldsymbol{V}_0|$ is the characteristic speed of the typical macroscopical length L_0.

The length l_D where viscous effects become comparable with non-linear transfer is used to define the velocity gradients: $|s| \approx \delta v(l_D)/l_D$ and it is determined by imposing that the effective Reynolds number on scale l_D is equal to unity, i.e.

$$\frac{\delta v(l_D)\, l_D}{\nu} = 1 . \tag{8}$$

It follows that the dissipative scale l_D is itself a function of h,

$$l_D(h) \sim \left(\frac{\nu}{v_0}\right)^{1/(1+h)} . \tag{9}$$

In order to stop the cascade one has to require that the smallest singularity exponent $h_{\min} > -1$. The value $h_{\min} = 0$, however, seems more reasonable and it is consistent with experimental and numerical data. The velocity gradients s are therefore

$$|s| = v_0 l_D^{h-1} = v_0^{2/(1+h)} \nu^{(h-1)/(h+1)} . \tag{10}$$

The conditional PDF of the gradients restricted to the points belonging to S(h) is related to PDF $\Pi(v_0)$ of the characteristic velocity difference v_0 on large scales by

$$P_h(s) = \Pi(v_0) \left|\frac{\mathrm{d}v_0}{\mathrm{d}s}\right| . \tag{11}$$

It follows that

$$P_h(s) \sim \left(\frac{\nu}{|s|}\right)^{(1-h)/2} \exp\left(-\frac{\nu^{1-h}|s|^{1+h}}{2\langle v_0^2\rangle}\right) . \tag{12}$$

In the K41 theory $h = 1/3$ uniformly in the fluid and

$$P(s) \sim \left(\frac{\nu}{|s|}\right)^{1/3} \exp\left(-\frac{\nu^{2/3}|s|^{4/3}}{2\langle v_0^2\rangle}\right) . \tag{13}$$

On the other hand, the probability of picking up a gradient singularity h is given by

$$\mathcal{P}_l(h)\,\mathrm{d}h \sim l^{3-D(h)}\,\mathrm{d}h\,. \tag{14}$$

The final PDF is

$$P(s) = \int \mathrm{d}h\, P_h(s)\, \mathcal{P}_{l_\mathrm{D}}(h) \sim \int \mathrm{d}h \left(\frac{\nu}{|s|}\right)^{2-[h+D(h)]/2} \times \exp\left(-\frac{\nu^{1-h}|s|^{1+h}}{2\langle v_0^2\rangle}\right). \tag{15}$$

Fig. 2 shows the PDF form (15) where $D(h)$ is the Legendre transformation (5) of (4) compared to the numerical data obtained in the shell model.

6. A mechanism for intermittency in the cascade model

Parisi [15] has recently proposed a mechanism, based on the existence of *soliton*-like solutions, to explain the intermittent behavior in the cascade model.

In the limit of zero viscosity and in absence of forcing, the shell model equations admit special solutions of the form

$$u(k,t) = k^{-h} f_h(y) \qquad \text{with } y = (t_0 - t)k^{\omega}, \tag{16}$$

where the function f_h goes to zero both at $+\infty$ and $-\infty$ and the exponents h and ω are related by $h = 1 - \omega$; the case $h = 1/3$ and $f = \text{const.}$ corresponds to K41

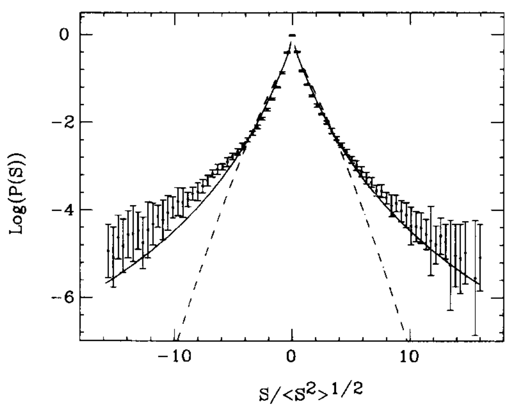

Fig. 2. Log–linear plot of the PDF of the gradients $P(s)$ versus s/σ, where $\sigma^2 = \langle s^2\rangle$. $\langle v_0^2\rangle = 10^{-2}$, $\nu = 10^{-6}$ and $N = 19$. The full line is the multifractal prediction given by eq. (15). $D(h)$ is given by fig. 1 through eqs. (4) and (5). Dashed line is the K41 prediction, dots are the numerical data.

theory. An approximated solution of eqs. (2) might be given by a linear superposition of the form:

$$u(k, t) = \sum_i k^{-h_i} f_{h_i}(y) \qquad \text{with } y = (t_i - t)k^{\omega_i} , \tag{17a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} k_i(t) = k_i(t)^{1+\omega_i} . \tag{17b}$$

(17) should be valid only in the region where different solutions do not overlap, i.e. where the baricenters $k_i(t) = |t_i - t|^{-1/\omega_i}$ remain separated. In order to satisfy the multifractal scaling [10], the interactions between "*solitons*" should lead to an equilibrium probability of finding an h-like solution given, for large k, by

$$P(k, h) \propto k^{-4+D(h)} . \tag{18}$$

It is interesting to understand whether it is possible to derive the power law (18) in this approximation. Indeed, (17b) shows that the different scale invariant solutions (strongly localized in wave number space) can be considered as a diluted gas of particles of coordinate k_i, each one escaping to infinity with different increasing speed. It follows that the collision of two particles labelled by singularities h_1, h_2 generates two new particles of type h_3, h_4, with a transition probability $P(h_1, h_2, h_3, h_4)$. The particles carry energy from the low wave number region to the large wave number region and a quasi-equilibrium distribution is obtained at large momenta as consequence of collisions.

In this picture, one would map the original differential equations to an interacting particle model where the transition probability, and then the $D(h)$ spectrum, might be computed analytically, e.g. by renormalization group techniques.

7. Conclusions

We have presented some results obtained with a dynamical system with few degrees of freedom which models the energy cascade in three-dimensional fully developed turbulence. The system shares a series of common properties with the phenomenology of fully developed turbulence, such as the anomalous scaling of the velocity difference and the probability distribution function of velocity gradients. These results are connected to the temporal intermittency of the chaotic time evolution, in the context of our system. The instantaneous Lyupanov exponent has very large fluctuations, strongly correlated with the energy dissipation at the Kolmogorov scale and with the localization of the instability along the dissipative modes. Our results and the recent work of Parisi suggest that this dynamical system with relatively few degrees of freedom can be useful to understand the multifractal nature of turbulence.

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