

Dynamical scaling and intermittency in shell models of turbulence

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We introduce a model for the turbulent energy cascade aimed at studying the effect of dynamical scaling on intermittency. In particular, we show that by slowing down the energy-transfer mechanism for fixed energy flux, intermittency decreases and eventually disappears. This result supports the conjecture that intermittency can be observed only if energy is flowing towards faster and faster scales of motion.

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Turbulent flows show very different behavior at changing the embedding physical dimension. The most spectacular change is in the reversal of the energy flux, from large to small scales in three dimensions and viceversa in two dimensions [1]. Moreover, the three-dimensional (3D) forward energy flux is strongly intermittent while the two-dimensional (2D) inverse energy transfer is almost Gaussian [2]. The presence of strong anomalous fluctuations in the 3D flux is believed to be connected to the existence of a hierarchical organization in the eddy turnover times, τ_r , at different scales. Simple dimensional arguments predict the eddy turnover time of velocity fluctuations at scale r in the inertial range to be of the order of $\tau_r \sim r^{2/3}$. For three-dimensional turbulence, the scenario is the one of an energy cascade from slow (large eddies) to fast (small eddies), without the possibility for fluctuations at different scales to *equilibrate*. This mechanism should be at the origin of the burstlike structure in the forward energy-transfer mechanism: any unusual fluctuation in the energy content at a given scale propagates to smaller and smaller scales until it is reabsorbed by a viscous mechanism. Hence, this is the cause of strong inhomogeneity in the spatiotemporal statistical properties of the energy-transfer process. The same qualitative arguments capture the absence of intermittency in the inverse 2D process, where energy flows from fast to slow modes, allowing the receiving modes to feel only the *mean* fluctuations of the unstable "mother" eddy. Intermittency in the related problem of scalar passive/active quantities advected by turbulent flows has been connected to the existence of a "dissipative anomaly," i.e., the presence of a nonvanishing scalar dissipation in the limit of small molecular diffusivity. Intermittency in the latter case has a natural Lagrangian interpretation connected to the existence of particles which separate even if starting in coinciding points [3,4]. Such a mechanism is absent in inverse cascade, pointing toward the conclusion that inverse cascade regimes cannot be intermittent. Here we intend to investigate a similar issue on a purely Eulerian base.

Intermittent fluctuations are usually related to the scaling properties of the Navier-Stokes equations [1]. Let us denote by $\mathbf{v}(\mathbf{x}, t)$ the velocity field of a turbulent flow satisfying the Navier-Stokes equations

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \pi + \nu \Delta \mathbf{v} + \mathbf{f}, \quad (1)$$

where $\pi = p/\rho$ (p being the pressure and ρ the density), ν is the kinematic viscosity of the flow, and \mathbf{f} is the (large scale)

external forcing. Equation (1) shows the remarkable property to be invariant under the scale transformation

$$\mathbf{x} \rightarrow \lambda \mathbf{x}, \quad \mathbf{v} \rightarrow \lambda^h \mathbf{v}, \quad t \rightarrow \lambda^{1-h} t, \quad \nu \rightarrow \lambda^{1+h} \nu. \quad (2)$$

Note that (2) implies for the energy dissipation: $\epsilon \rightarrow \lambda^{3h-1} \epsilon$, i.e., ϵ is constant if $h=1/3$ as predicted by the K41 theory. In the multifractal theory of turbulent flows, h is supposed to be a fluctuating quantity, although on the average the rate of energy dissipation is constant [1]. The above scenario is complemented by the assumption that the statistical fluctuations are described by a scaling-invariant probability distribution $P_h \sim \lambda^{3-D(h)}$, where $D(h)$ can be interpreted as the fractal dimension related to the fluctuations λ^h . Regardless of the geometrical interpretation of $D(h)$, the basic physical questions are which is the mechanism determining the fluctuations of h and why are there fluctuations in the energy flux. The above, scale-invariant, scenario of the nonequilibrium statistical properties of turbulent flows suggests a simple although not trivial picture of intermittency, related to the energy cascade mechanism. At scale r , the amount of kinetic energy due to turbulent fluctuations can be estimated as $(\delta_r v)^2$, where $\delta_r v = v(x+r) - v(x)$, and where we neglect vectorial indexes for simplicity. We expect that the rate of energy flux at scale r , denoted by ϵ_r , is of the order of $(\delta_r v)^2 / \tau_r$. Because energy transfer is due to nonlinear interactions, τ_r can be estimated as $r / \delta_r v$. Thus we obtain $(\delta_r v)^3 \sim r \epsilon_r$ which, as expected, is invariant under (2). As a result, the energy-transfer statistics are strongly correlated to the fluctuations on the energy contents at different scales, $(\delta_r v)^2$, and to their dynamical properties, τ_r . Moreover, the presence of strong intermittent fluctuations in the energy contents at different scales, reflects, via the equation of motions, into nontrivial fluctuations of the local eddy turnover times. Indeed, previous theoretical and numerical works have demonstrated that spatial and temporal properties of the energy cascade mechanism are strongly correlated [5,6].

The previous phenomenological arguments suggest that intermittency can be observed only if energy is flowing towards faster scales of motion. It is therefore tempting to argue that by decreasing the scaling exponents of the eddy turnover times along the cascade, i.e., by slowing down the energy-transfer mechanism for fixed energy flux, intermittency should decrease and eventually disappear. Our aim in

this paper is to provide clear evidence that the above conjecture holds.

We investigate the case of a shell model of turbulence (see [1,7,8] and references therein), where the time needed to transfer a fluctuation from a scale R to a scale r with $r < R$ is given by the estimate: $\tau_{R,r} \propto R^{x-1/3} - r^{x-1/3}$, where the usual case of K41 turbulence is described by $x=2/3$. We will show that in the limit $x \rightarrow 1/3$, i.e., when the eddy turnover times at different scales become of the same order, the energy transfer becomes less and less intermittent. This finding supports the idea that intermittency is generated by the impossibility for fluctuations at a given scale to be reabsorbed before being transferred down scale.

The motivation to use shell models as a possible surrogate of the Navier-Stokes dynamics is twofold. First, shell models proved to be very successful in reproducing many of the statistical features of both 2D and 3D turbulent flows, being at the same time much easier to simulate numerically. Second, they are flexible enough to allow a structural change in their equations of motion which will allow us to directly probe the importance of time dynamics in fixing the intermittent properties of the energy-transfer process (see below). Moreover, shell models describe the evolution of turbulent flows in a Lagrangian or quasi-Lagrangian reference frame, i.e., where sweeping of small scales by large scales are absent [9,10]. This allows us to study the true “built-in” dynamical properties of the energy cascade without the overwhelming effects induced by the large scale flow.

In a shell model, the basic variable describing the “velocity field” at scale $r_n = \Lambda^{-n} r_0 \equiv k_n^{-1}$, is a complex number u_n satisfying a suitable set of nonlinear equations. There are many versions of shell models that have been introduced in literature (see [8] for a recent review). Here we choose the one proposed in [11], which is an improved version of the so-called GOY model [12,13]

$$\begin{aligned} \frac{du_n}{dt} = & ik_n [\Lambda u_{n+1}^* u_{n+2} + b u_{n-1}^* u_{n+1} - c \Lambda^{-1} u_{n-2} u_{n-1}] \\ & - \nu k_n^2 u_n + f_n, \end{aligned} \quad (3)$$

where $\Lambda=2$, $c=-(1+b)$, and f_n is an external forcing. In shell models, we can associate δv with u_n . Clearly Eq. (3) satisfies the scaling (2). The important point on shell models like (3) is that the statistical properties of intermittent fluctuations, computed either using u_n or the instantaneous rate of energy dissipation, are in close *qualitative* and *quantitative* agreement with those measured in laboratory experiments, for homogeneous and isotropic turbulence. Thus, shell models provide a useful tool to investigate in a simple way the physical consequences of scaling (2) and intermittency. Moreover, at variance from 3D Navier-Stokes equations, the computational complexity grows with Reynolds only as $\text{Re}^{1/2}$, allowing for reliable numerical studies also at very high Reynolds numbers.

It is easy to realize that also in shell models, $\tau(k_n)$ goes as k_n^{h-1} as predicted by (2), i.e., the characteristic time for energy transfer decreases quite fast as the scale k_n^{-1} is decreasing. If $\tau(k_n)$ is independent of k_n , the constant-flux argument would suggest energy equipartition among all scales, i.e.,

$(\delta v)^2 \sim \epsilon_r$. To be more specific, by constraining ourselves on shell models, let us modify (3) in such a way that there still exists an average Reynolds independent rate of energy dissipation $\epsilon = \langle \nu \sum_n k_n^2 |u_n|^2 \rangle$ while the statistical properties are invariant under the scaling transformation

$$k_n \rightarrow \lambda^{-1} k_n, \quad u_n \rightarrow \lambda^h u_n, \quad t \rightarrow \lambda^{-x} t, \quad \nu \rightarrow \lambda^{1+h} \nu, \quad (4)$$

where x is a fixed parameter in the equations. Note that for Eq. (4) to hold, we should require that the nonlinear terms scale as λ^{2h-x} . Note also that $\epsilon \sim \nu (\nabla v)^2 \sim \lambda^{3h-1}$, thus we do not change the constrain on the energy flux.

In order to satisfy the scaling (4), we introduce the following equations:

$$\begin{aligned} \frac{du_n}{dt} = & ik_0^{1-x} k_n^x \left[2u_{n+1}^* u_{n+2} + b u_{n-1}^* u_{n+1} + \frac{1+b}{2} u_{n-2} u_{n-1} \right] \\ & - \nu k_n^{1+x} k_0^{1-x} u_n + f_n, \end{aligned} \quad (5)$$

where k_0 is the largest scale in the system. Clearly, we obtain the old shell model for $x=1$, while a direct inspection of (5) shows that (4) is satisfied for any x . Moreover, for $\nu=0$, the generalized energy, $Q = k_0^{x-1} \sum_n k_n^{1-x} |u_n|^2$, is conserved by nonlinear interactions. Thus, in a statistically stationary regime we have the “energy” budget

$$\frac{1}{2} \frac{dQ}{dt} = 0 = P - \nu \sum_n k_n^2 |u_n|^2,$$

where $P \propto \text{Re}(\langle \sum_n k_n^{1-x} u_n^* f_n \rangle)$ is the energy input. Note that in this shell model energy fluctuations at scale k_n are of the order $Q_n = k_0^{x-1} k_n^{1-x} |u_n|^2$ while the eddy turnover time for the energy transfer is of the order

$$\tau(k_n) \equiv 1/(k_0^{1-x} k_n^x u_n). \quad (6)$$

Thus the energy flux through wave number k_n can be proved rigorously to be of the order of $\Pi_n = k_n |u_n|^3$ as for the original shell model. Therefore, for all x , by keeping $\langle \Pi_n \rangle = \text{const}$ we can achieve a constant energy flux,

$$\langle |u_n|^3 \rangle \sim k_n^{-1}. \quad (7)$$

In summary, we want to stress that for all x the shell model (5) reproduces the same average energy flux and maintains the same relative weights between the nonlinear terms appearing in the right-hand side of (5). The only aspect that changes is the dynamical efficiency in transferring energy between shells. In particular, for small x , say $x \sim 1/3$, by combining (6) and (7) we should expect all eddy turnover times to be of the same order in the inertial range and therefore the energy to reach a quasiequipartition state for fixed value of ϵ . The shell model provides us with the interesting possibility to study intermittency as a function of x , i.e., as a function of the *dynamical* scaling (4) and (6). Intuitively, one can imagine that decreasing x from its Navier-Stokes value $x=1$ induces a smoother and smoother energy-transfer process towards smaller scales, fluctuations among different scales tend to equilibrate each other and, consequently, non-Gaussian fluctuations are depleted. Thus, as a function of x the shell model (5) should exhibit a kind of phase transition from “strong” intermittent fluctuations at $x=1$ to Gaussian

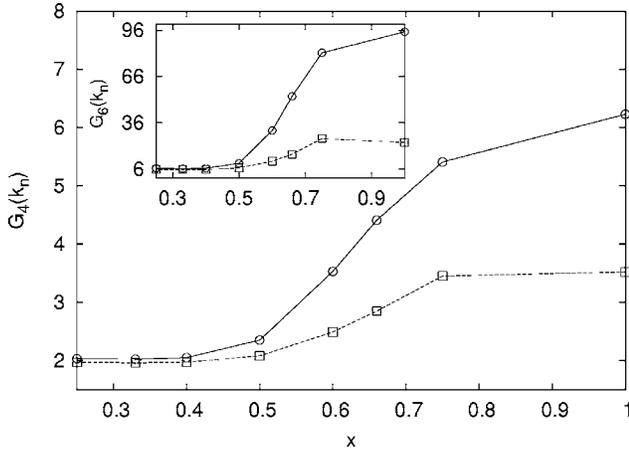


FIG. 1. The value of $G_{2p}(k_n) = \langle |u_n|^{2p} \rangle / \langle |u_n|^2 \rangle^p$ for $p=2$ and $p=3$ (in the inset) as a function of x and for two different values of k_n , namely $n=10$ (boxes) and $n=15$ (circles). The predicted value for the Gaussian statistics (horizontal dotted line) is reported for comparison.

nonintermittent fluctuations at x small. In this paper we support the above conjecture by numerical simulations of the model (5). Moreover, we will give a theoretical argument to estimate the value of x_c below which intermittency starts to be depleted.

In order to keep ϵ constant, regardless of the value of ν and x , we use the forcing $f_n = A_n / u_n^*$ for $n=1, 2$ and $f_n=0$ for $n > 2$. We have performed a series of numerical simulations for different values of x , keeping the parameter $b = -0.4$ fixed. As a measure of intermittency, we compute the generalized kurtosis $G_{2p}(k_n) = \langle |u_n|^{2p} \rangle / \langle |u_n|^2 \rangle^p$. All generalized kurtosis possess scaling properties, $G_{2p}(k_n) \sim k_n^{-\zeta(2p) + p\zeta(2)}$, where the $\zeta(p)$ are the scaling exponents of the p th-order structure functions. The absence of intermittency, $\zeta(2p) = p\zeta(2)$, implies that all generalized kurtosis become independent of k_n and equal to their large-scale quasi-Gaussian behavior. Figure 1 shows the value of $G_4(k_n)$ for different values of k_n as a function of x , while in the inset we show the same but for $G_6(k_n)$. As one can clearly see, for x smaller than 0.5 a sharp decrease of G_4 and G_6 is observed. For small values of x both G_4 and G_6 become equal to their Gaussian value. The above results tell us that intermittency is depleted for small x in agreement with our intuitive argument. Also, within the numerical error bars, for $x > x_c \sim 0.5$, intermittent fluctuations seem to weakly depend on x .

We want now to understand why the transition to non-intermittency fluctuations is observed for relatively large value of x , namely for $x \sim 0.4$. As we shall see in the following, $x_c = 1/3 + \Delta x$, where Δx is due to intermittency correction. The argument goes as follows. Let us consider two scales, k_n and k_{n+m} . The corresponding times for the energy transfer are $\tau(k_n) \sim k_n^{h'-x}$ and $\tau(k_{n+m}) \sim k_{n+m}^{-x}$, where h and h' are the values of the ‘‘local’’ fluctuations of $u_n \sim k_n^{-h}$ and $u_{n+m} \sim k_{n+m}^{-h}$, respectively. For $x=1$ the probability that the ratio $\chi_m = \tau(k_{n+m}) / \tau(k_n)$ is larger than 1 is extremely small, already for $m=1, 2$. When x becomes smaller than 1 the probability $P(\chi_m > 1)$ starts growing, even for small m . If χ_m

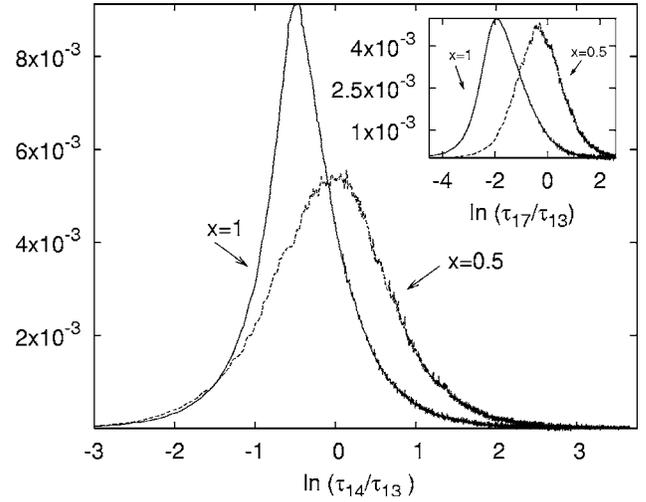


FIG. 2. Probability distribution of $\ln[\tau(k_{14})/\tau(k_{13})]$ for two different values of x , namely $x=1$ and $x=0.5$. The theory presented in the paper predicts that intermittency is depleted at $x=0.5$ because $\tau(k_{n+1}) \sim \tau(k_n)$. In the inset we show the probability distribution of $\ln[\tau(k_{17})/\tau(k_{13})]$.

is much larger than 1, then the energy transfer from scale k_n to scale k_{n+m} is stopped and energy tends to a quasiequipartition state. In order to estimate x_c , let us compare, for a given k_n , the ratio between two eddy turnover times $T_m(q) = \langle (\chi_m)^q \rangle$, in the limit of large-scale separation,

$$T_m(q) \sim \left(\int dh k_m^{[qh - qx - 3 - D(h)]} \right) \sim k_m^{[-qx - \zeta(-q)]},$$

$$\zeta(q) = \inf_h [qh + 3 - D(h)], \quad (8)$$

where the result is obtained in the saddle-point limit $k_m \rightarrow \infty$ by considering the fluctuations at scale k_n and k_{n+m} almost uncorrelated (random multiplicative cascade).

The above expression tells us that $T_m(q)$ becomes larger and larger, for $k_m \rightarrow \infty$, when $x < x_c(q) = -\zeta(-q)/q$. Let us note that, because of convexity properties of $\zeta(q)$, $x_c(q)$ is an increasing function of q . Thus, there is not a single value of x below which intermittency is depleted, rather the transition to quasi-Gaussianity is continuous; different moments of the eddy turnover ratios behave in slightly different way. As a simple guideline the transition region is $[x_c(1), x_c(2)] = [0.4, 0.42]$, estimated by using the $D(h)$ curve which fits the $\zeta(q)$ exponents as given by the She-Leveque formula [14]. A direct numerical investigation gives further support to our previous argument. In Fig. 2 we show the probability density function of the ratio $\tau(k_{14})/\tau(k_{13})$, with $\tau(k_n)$ being the instantaneous eddy turnover time at shell k_n . For $x=1$, we expect that $\tau(k_{14}) < \tau(k_{13})$ with probability close to 1, while, according to our estimate, we expect that for $x=0.5$ $\tau(k_{14}) \sim \tau(k_{13})$ and intermittency is depleted. This is clearly shown in Fig. 2, where in the inset we also show the probability density function of $\ln[\tau(k_{17})/\tau(k_{13})]$. Hence, we can con-

clude that for all scales k_n the eddy turnover times become almost equal producing a quasi-Gaussian distribution in the generalized energy.

As a further check of our argument, we consider the shell model (5) for $b=-0.8$. In this case, the model at $x=1$ shows larger intermittent correction with respect to the case $b=-0.4$ previously considered. According to our argument for the transition to occur, we should expect that for $b=-0.8$, depletion of intermittency takes place for larger values of x , which is indeed the case as shown in Fig. 3.

In summary, we have presented the following main results: (i) we introduced a new version of the shell model which satisfies a generalization of the dynamical scaling (4); (ii) we have proposed a simple, although nontrivial, argument for understanding how intermittency can depend on the scaling properties of the eddy turnover time; (iii) we have shown, by numerical simulations, that our argument is correct; (iv) we have provided a multifractal estimate of the critical value, x_c , where intermittency should disappear.

This study can be considered a natural extension of a series of previous works meant to stress the importance of temporal correlation in determining the spatial anomalous fluctuation in turbulent flows and turbulent models [15,16] (see also [5] for similar consideration for Navier-Stokes equations.)

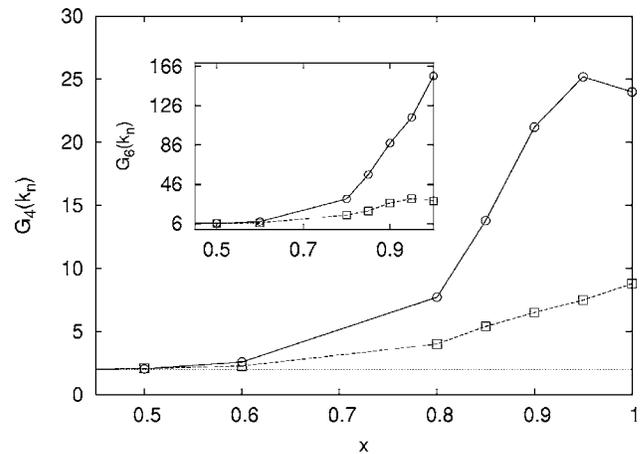


FIG. 3. Same quantities as in Fig. 1 for the model with parameter $b=-0.8$ with $n=10$ (boxes), $n=15$ (circles) in the main figure, and $n=5$ (boxes), $n=9$ (circles) in the inset. The increase of intermittency at $x=1$ increases the critical value of x for which a *quasi-Gaussian* distribution is attained.

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