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Lattice Boltzmann Method for Fluid Flows on Spherical Surfaces

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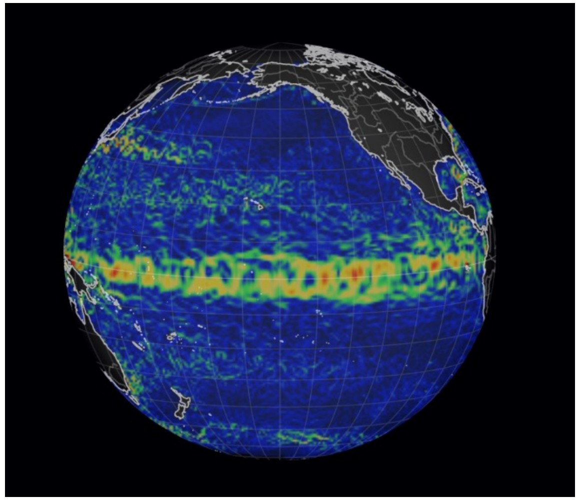
Introduction

Motivation

Soap bubble

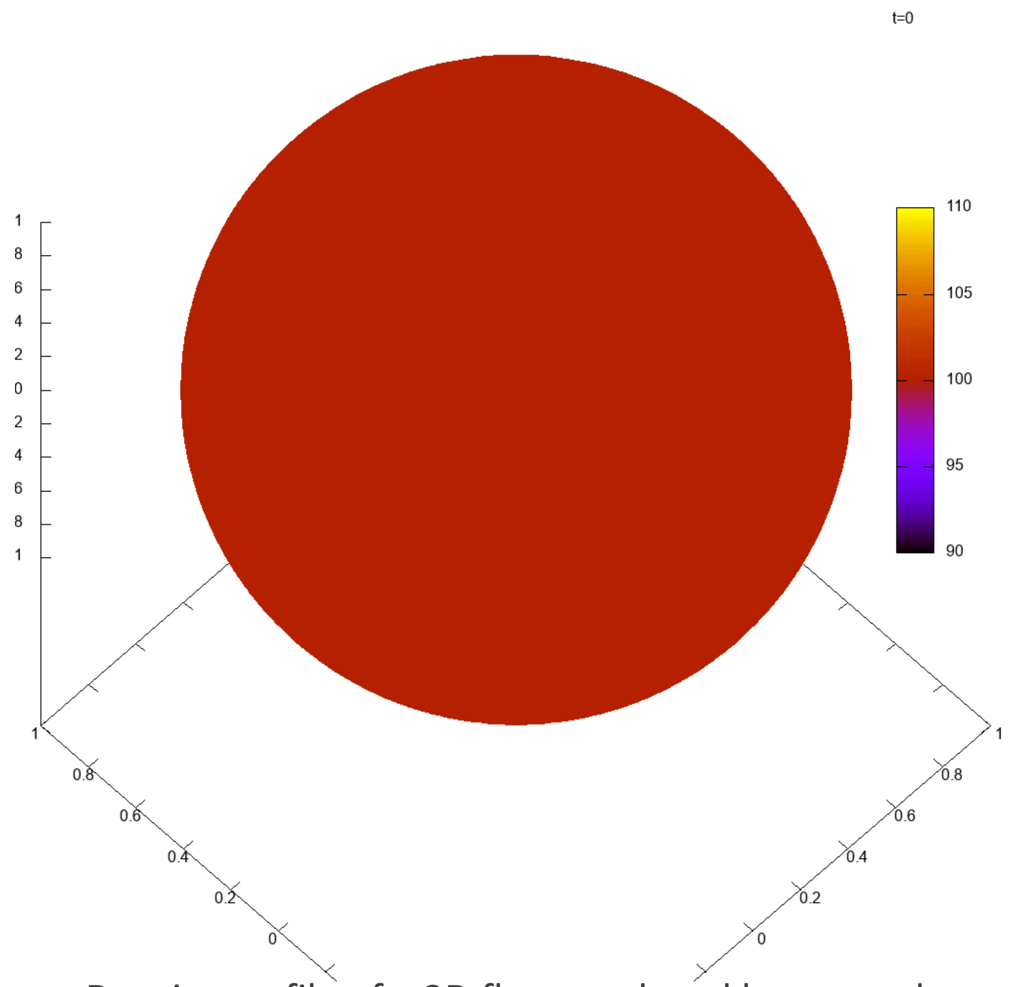
- Spatial curvature leads to the observation of peculiar flow phenomena
- Numerical simulations of fluids over curved media allow to describe such phenomena



Waves trapped at the Earth’s equator

Approach

We developed a Lattice Boltzmann method (LBM) suitable to deal with curvilinear coordinates for the spherical surface



Density profile of a 2D flow produced by our code

Outline

- Extension of LBM to spherical surface using vielbein formalism
- Formulation of (axisymmetric) benchmark problems for the spherical surface
- 2D flows simulations on the spherical surface

Methods

Vielbein Field

Construction of a vector field on the spherical surface as a non-coordinate basis (*vielbein* formalism)

$$e_{\hat{a}} = e^a_{\hat{a}} \partial_a, \qquad \mathbf{u} = u^{\hat{a}} e_{\hat{a}}$$

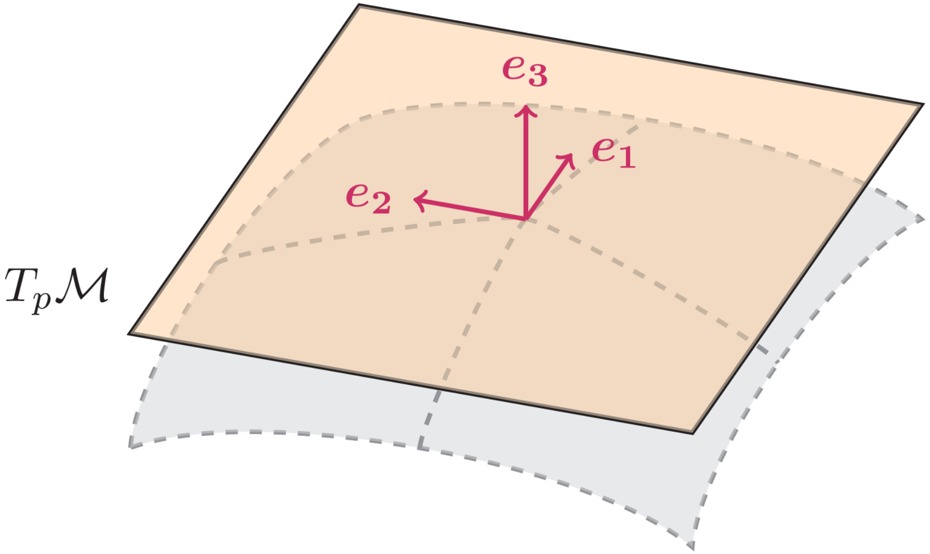
$$g_{ab} e^a_{\hat{a}} e^b_{\hat{b}} = \delta_{\hat{a}\hat{b}}, \qquad u^2 = \delta_{\hat{a}\hat{b}} u^{\hat{a}} u^{\hat{b}}$$

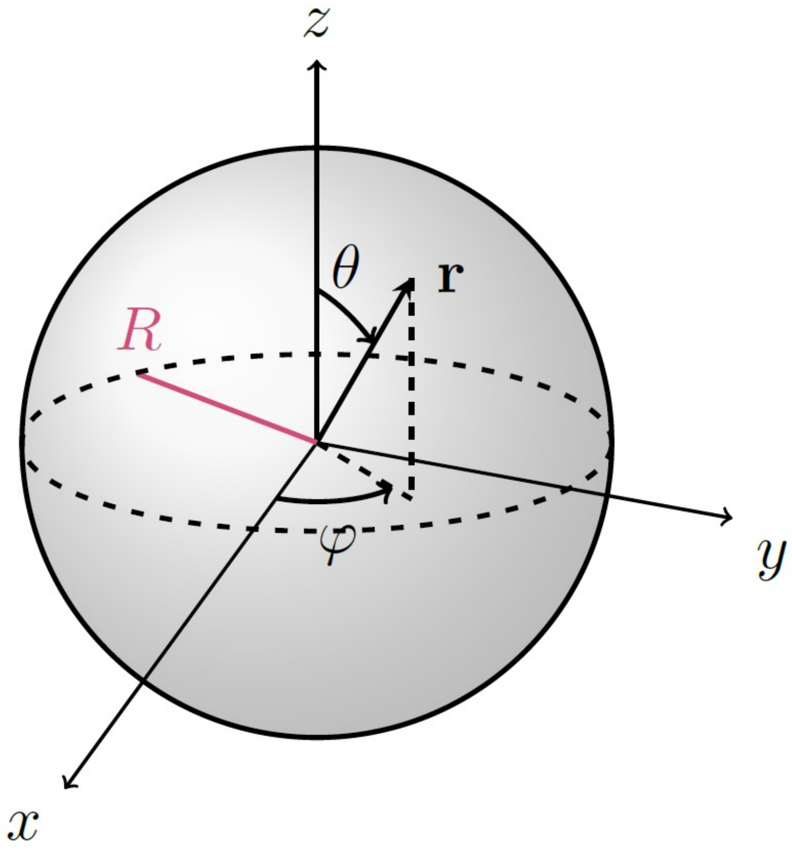
$$\omega^{\hat{a}}_a e^a_{\hat{b}} = \delta^{\hat{a}}_{\hat{b}}, \qquad \omega^{\hat{a}}_a e^b_{\hat{a}} = \delta^b_a$$

$$e^\theta_{\hat{\theta}} = \frac{1}{R}, \qquad e^\varphi_{\hat{\varphi}} = \frac{1}{R\sin\theta}$$

$$\omega^{\hat{\theta}}_\theta = R, \qquad \omega^{\hat{\varphi}}_\varphi = R\sin\theta$$

$$\Gamma^{\hat{\theta}}{}_{\hat{\varphi}\hat{\varphi}} = -\Gamma^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}} = -\frac{\cos\theta}{R\sin\theta}$$





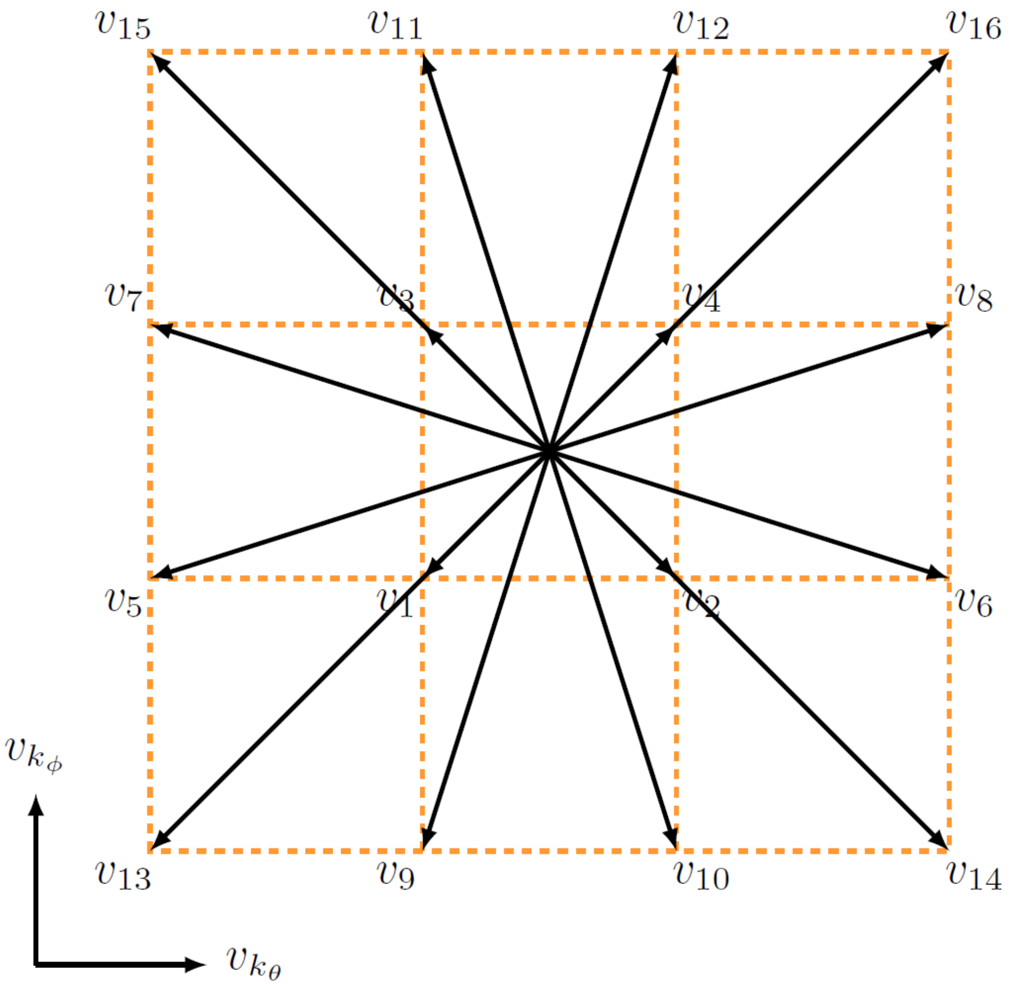
D2Q16 Velocity Stencil

Gauss-Hermite quadrature of order 4:

$$\rho(\mathbf{x},t) = m\sum_{k=1}^{16} f_k$$

$$\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t) = m\sum_{k=1}^{16} f_k\mathbf{v}_k$$

$$\rho(\mathbf{x},t)e(\mathbf{x},t) = \frac{m}{2}\sum_{k=1}^{16} f_k|\mathbf{v}_k - \mathbf{u}|^2$$



Boltzmann Eq on the Spherical Surface

- Cartesian:

Advection, Collisions

$$\frac{\partial f}{\partial t} + v^x \frac{\partial f}{\partial x} + v^y \frac{\partial f}{\partial y} + \frac{F^x}{m}\frac{\partial f}{\partial v^x} + \frac{F^y}{m}\frac{\partial f}{\partial v^y} = -\frac{1}{\tau}(f - f^{\rm eq})$$

Advection, External forcing

$$\frac{1}{R\sin\theta_s}\left[\frac{\mathcal{F}^\theta_{s+\frac{1}{2},q} - \mathcal{F}^\theta_{s-\frac{1}{2},q}}{\delta\theta} + \frac{\mathcal{F}^\varphi_{s,q+\frac{1}{2}} - \mathcal{F}^\varphi_{s,q-\frac{1}{2}}}{\delta\varphi}\right]$$

$$\theta_s = \frac{\pi}{N_\theta}(s - 1/2) \qquad (1 \le s \le N_\theta)$$

$$\varphi_q = \frac{2\pi}{N_\varphi}(q - 1/2) \qquad (1 \le q \le N_\varphi)$$

Advection

- Covariant-vielbein:

Collisions

$$\frac{\partial f}{\partial t} + \frac{1}{R\sin\theta}\left[v^{\hat\theta}\frac{\partial(f\sin\theta)}{\partial\theta} + v^{\hat\varphi}\frac{\partial f}{\partial\varphi}\right] + \frac{\cos\theta}{R\sin\theta}\left[v^{\hat\varphi}\frac{\partial(f v^{\hat\varphi})}{\partial v^{\hat\theta}} - v^{\hat\theta}\frac{\partial(f v^{\hat\varphi})}{\partial v^{\hat\varphi}}\right] + \frac{F^{\hat\theta}}{m}\frac{\partial f}{\partial v^{\hat\theta}} + \frac{F^{\hat\varphi}}{m}\frac{\partial f}{\partial v^{\hat\varphi}} = -\frac{1}{\tau}(f - f^{\rm eq})$$

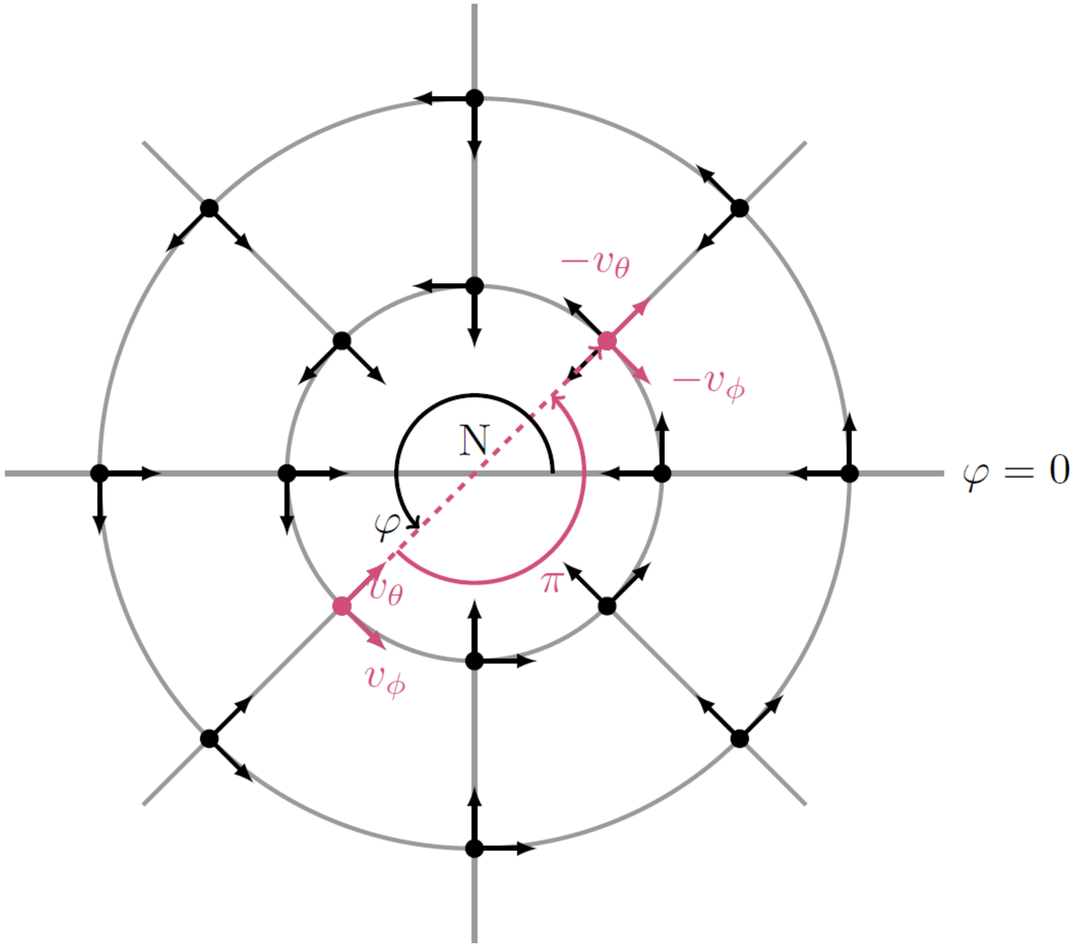
Advection, Internal forcing, External forcing

$$\Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} = -\Gamma^{\hat\varphi}{}_{\hat\theta\hat\varphi} = -\frac{\cos\theta}{R\sin\theta}$$

Advection on the Spherical Surface

- We employ finite-difference schemes (WENO-5)
- How to deal with the non-periodicity along theta?
- Populate the ghost nodes with:

$$f(-\delta\theta\,,\varphi;v^{\hat{\theta}},v^{\hat{\varphi}}) = f(+\delta\theta\,,\varphi+\pi;-v^{\hat{\theta}},-v^{\hat{\varphi}})$$
$$f(\pi+\delta\theta\,,\varphi;v^{\hat{\theta}},v^{\hat{\varphi}}) = f(\pi-\delta\theta\,,\varphi+\pi;-v^{\hat{\theta}},-v^{\hat{\varphi}})$$



Benchmark Problems

Shear Wave Damping

- Cartesian: $u^y = u^y(x)$ $$\frac{\partial u^y}{\partial t} = \nu \frac{\partial^2 u^y}{\partial x^2}$$
- Spherical surface: $u^{\hat\varphi} = u^{\hat\varphi}(\theta)$ $$\frac{\partial}{\partial t}\left(\frac{u^{\hat\varphi}}{\sin\theta}\right) = \underbrace{\frac{\nu}{R^2 \sin^3\theta}\frac{\partial}{\partial\theta}\left[\sin^3\theta \frac{\partial}{\partial\theta}\left(\frac{u^{\hat\varphi}}{\sin\theta}\right)\right]}_{\text{Eigenvalue Eq}}$$

$$u_0^{\hat\varphi}(\theta) = V_0$$

$$u^{\hat\varphi}(t,\theta) = \sin\theta \sum_{n=0}^{\infty}[A_n(t)\ \underbrace{F_n(\theta)}_{\text{Eigenfunc}}\ ]$$

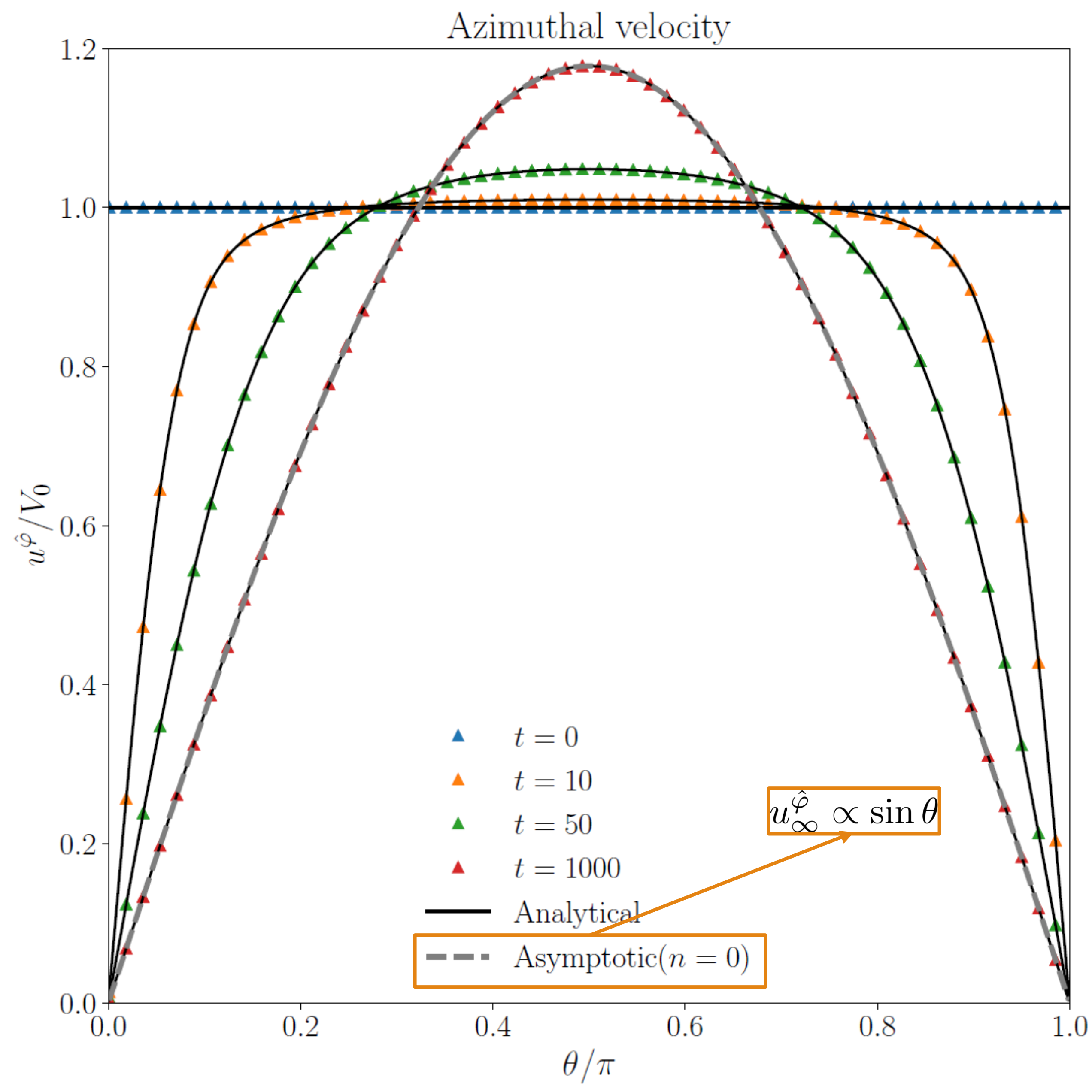
$$A_n(t) = a_n e^{-\alpha_{e;n} t}$$

$$\alpha_{e;n} = \frac{\nu}{R^2}\chi^2_{e;n} = \frac{\nu}{3}\underbrace{2n(2n+3)}_{\text{Eigenvalue}^2}$$

n	χ_n	$A_n(t)$	$F_n(\theta)$
0	0	$\pi\sqrt{3}/4$	$\sqrt{3/2}$
1	$\sqrt{10}$	$\pi\sqrt{21}e^{-\nu 10t/3R^2}/32\sqrt{2}$	$\sqrt{21/32}(3+5\cos 2\theta)$
2	$\sqrt{28}$	$\pi\sqrt{165}e^{-\nu 28tR^2}/256$	$\sqrt{165}(15+28\cos 2\theta+21\cos 4\theta)/128$

Shear Solution

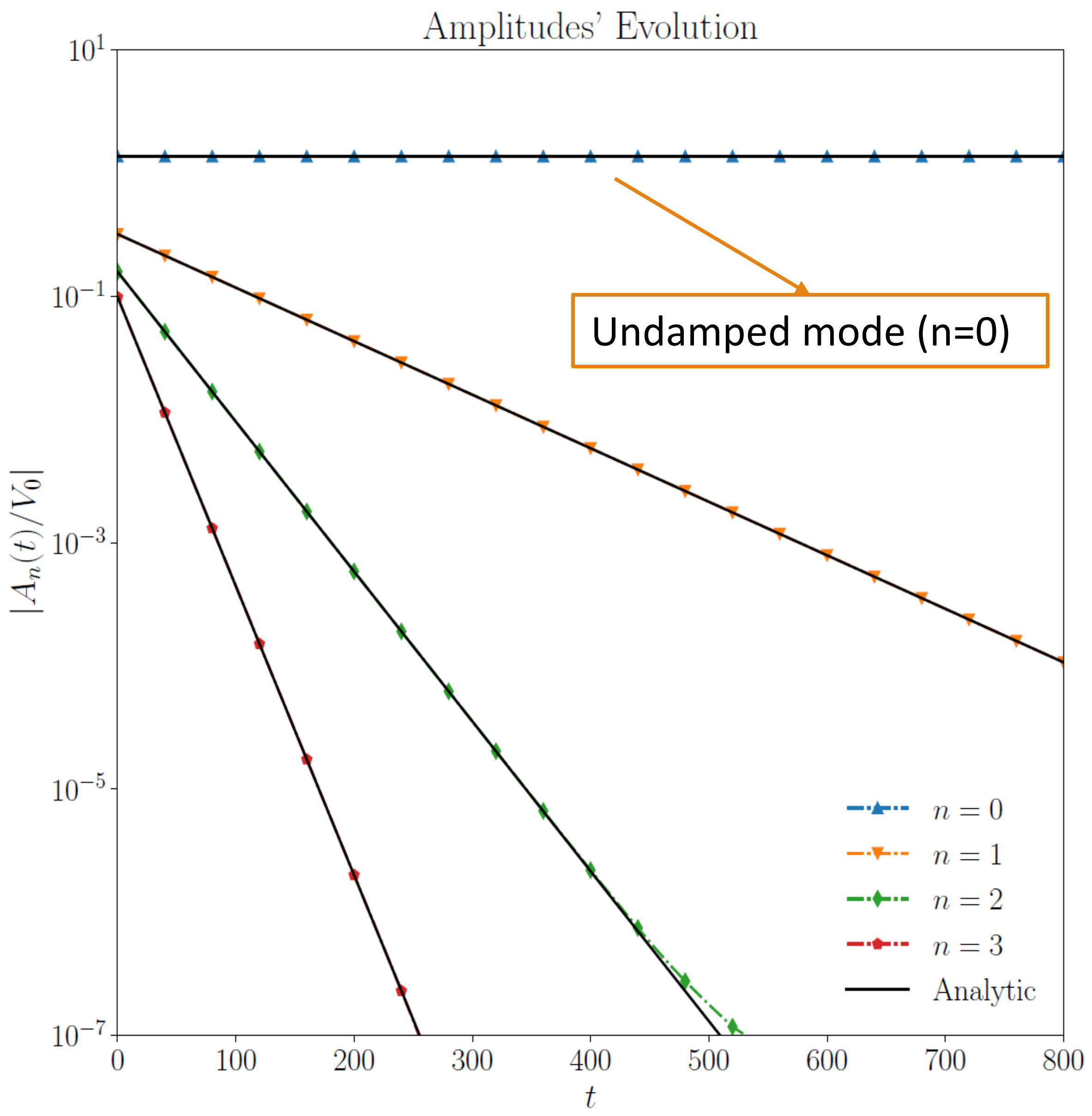
The velocity profile is accurately recovered at every *t* and converges to the expected asymptotic behaviour



1×256 grid, D2Q16,
$\delta t = 1\times10^{-4},\ \tau = 1\times10^{-3},\ R = 1$
(WENO-5)

Shear amplitudes

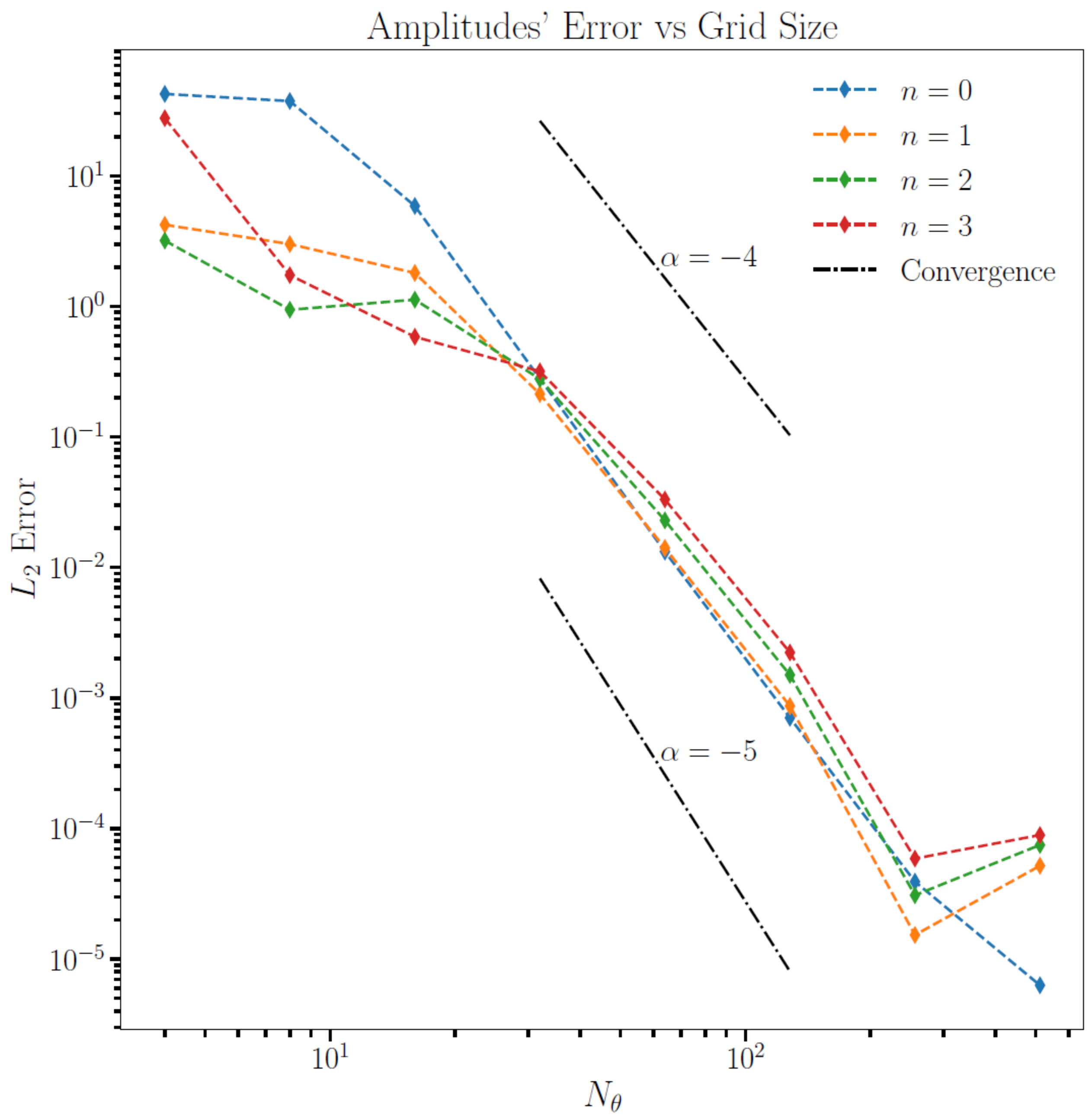
The n=0 mode experiences no damping



1×256 grid, D2Q16,
$\delta t = 1\times10^{-4},\ \tau = 1\times10^{-3},\ R = 1$
(WENO-5)

Shear wave convergence

The convergence is near fifth-order as expected



$8 \times N_\theta$ grid, D2Q16,
$\delta t = 1 \times 10^{-4},\ \tau = 1 \times 10^{-3},\ R = 1$
(WENO-5)

Sound wave propagation

We consider an initial velocity profile of the type: $u^{\hat{\theta}} = u^{\hat{\theta}}(\theta)$

Ideal fluid:

$$\frac{\partial^2 u^{\hat{\theta}}}{\partial t^2} = \underbrace{\frac{c_s^2}{R^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(u^{\hat{\theta}}\sin\theta\right)\right]}_{\text{Eigenvalue Eq}}$$

$$\underbrace{\lambda_n^2 = \sqrt{2n(n-1)} - 1}_{\text{Eigenvalues}}$$

Dissipative fluid:

$$\left(-iR^2\omega_n + 2\nu\right) u_n^{\hat{\theta}} = \left(\frac{ic_s^2}{\omega_n} + 2\nu\right)\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial(u^{\hat{\theta}}\sin\theta)}{\partial\theta}\right], \qquad \omega_n = -i\zeta_n + \alpha_n(\nu)$$

$$u^{\hat{\theta}}(t,\theta) = \frac{1}{\sin\theta}\sum_{n}^{\infty}\left[A_n(t) F_n(\theta)\right]$$

$$A_n(t) = a_n e^{-\zeta_n t}\cos[\alpha_n(\nu) t]$$

$$\zeta_n = \frac{\nu}{2R^2}(\lambda_n^2 - 1)$$

$$\alpha_n(\nu) = \sqrt{c_s^2\lambda_n^2/R^2 - \zeta_n^2}$$

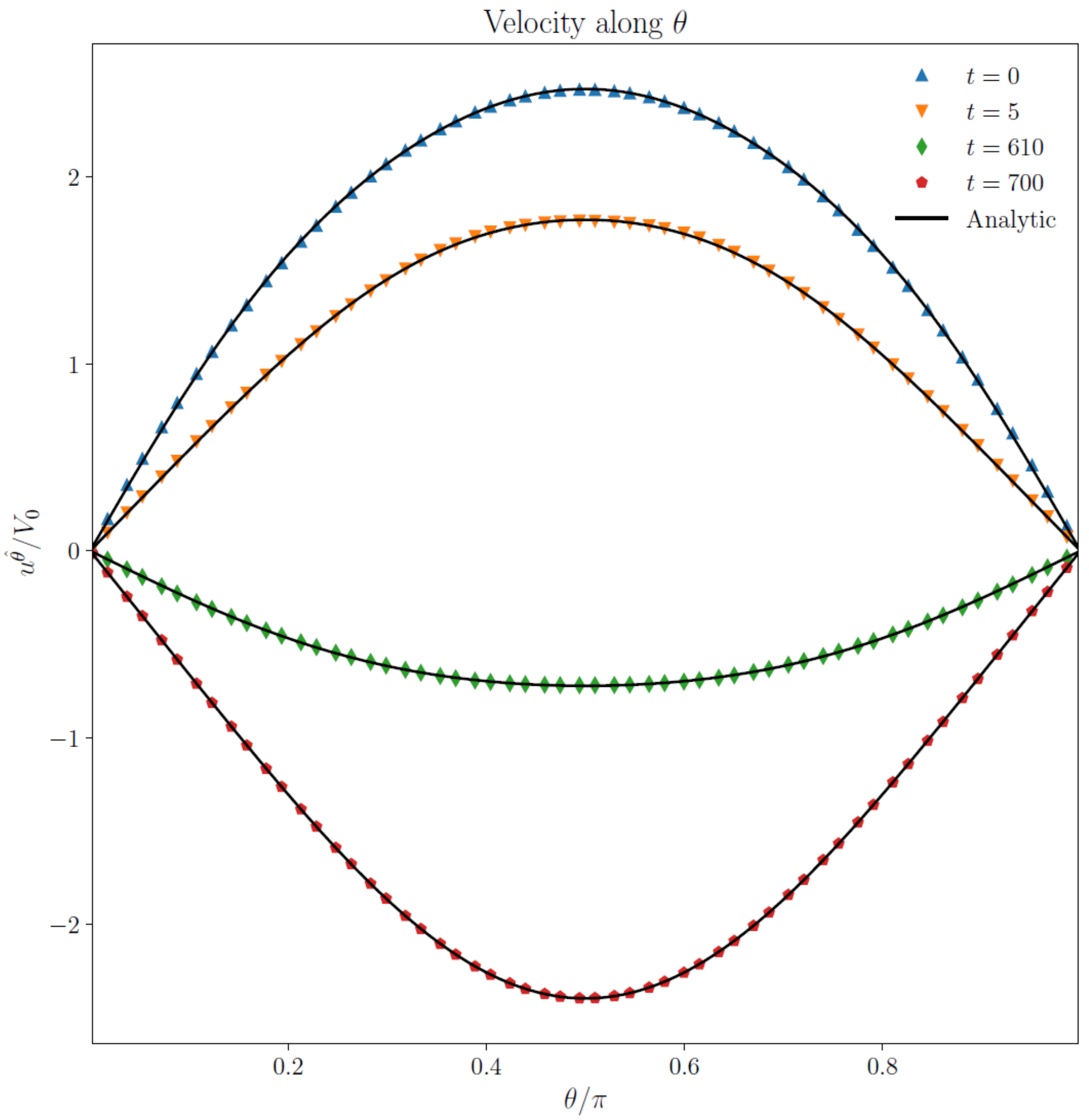
n	λ_n	a_n	$F_n(\theta)$
1	0	$\pi(\pi^2+3)/8\sqrt{3}$	$\sqrt{3/2}\sin\theta$
2	2	$-\pi\sqrt{7}\sqrt{6}(4\pi^2-33)/256$	$\sqrt{21/32}\sin^2\theta(1-3\cos^2\theta)$
3	$\sqrt{18}$	$\pi\sqrt{55}(4\pi^2-37)/2048\sqrt{3}$	$\sqrt{165}/16\sin^2\theta(1-14\cos^2\theta+21\cos^4\theta)$

Sound wave solution

The velocity profile of a sound wave in case of an axisymmetric flow is accurately recovered

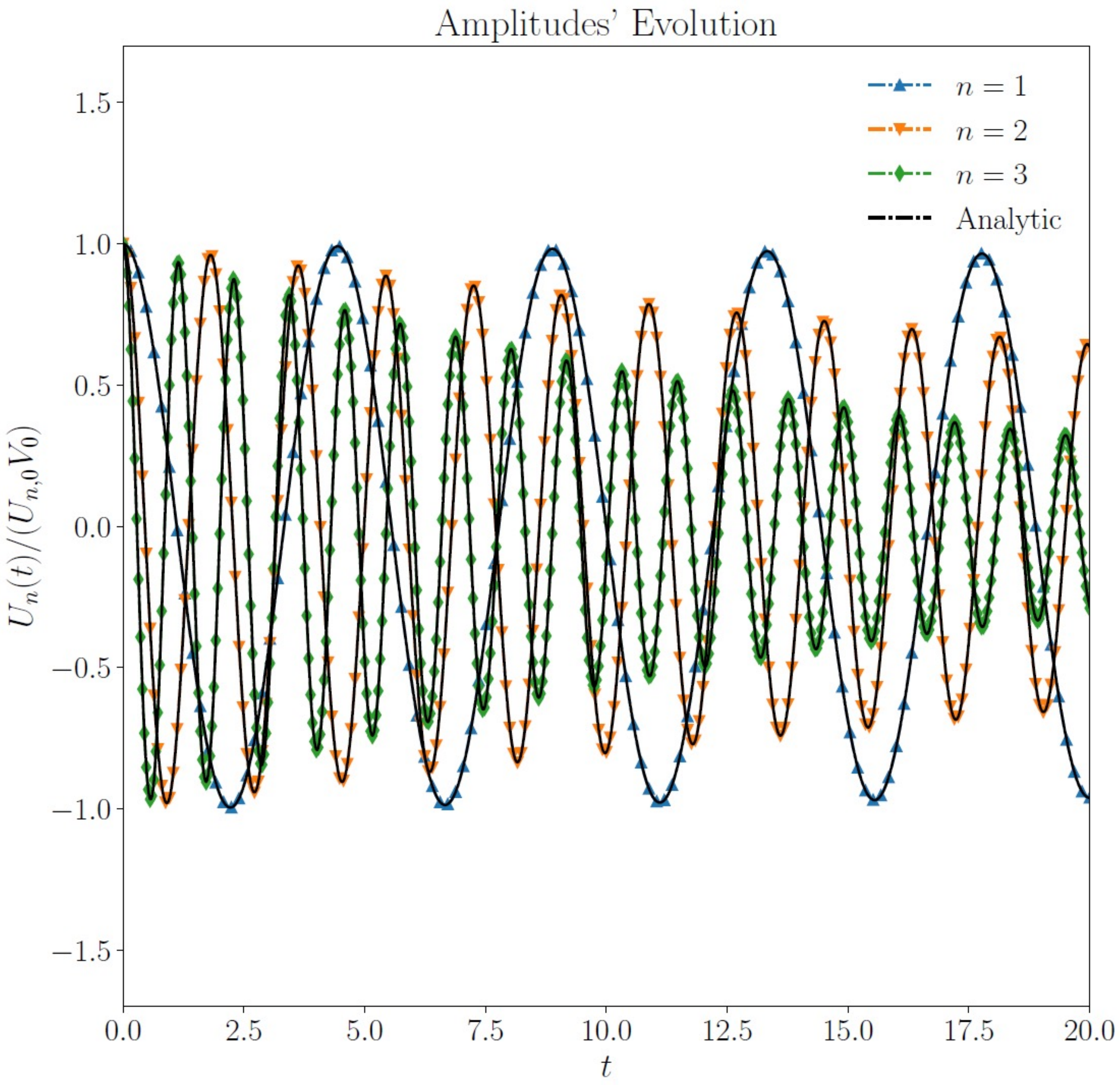
$$u_0^{\hat{\theta}}(\theta) = U\theta(\pi - \theta)$$

$1{\times}256$ grid,
$\delta t = 2{\times}10^{-5},\ \tau = 2{\times}10^{-5},\ R = 1$
(WENO-5)



Amplitudes' Evolution

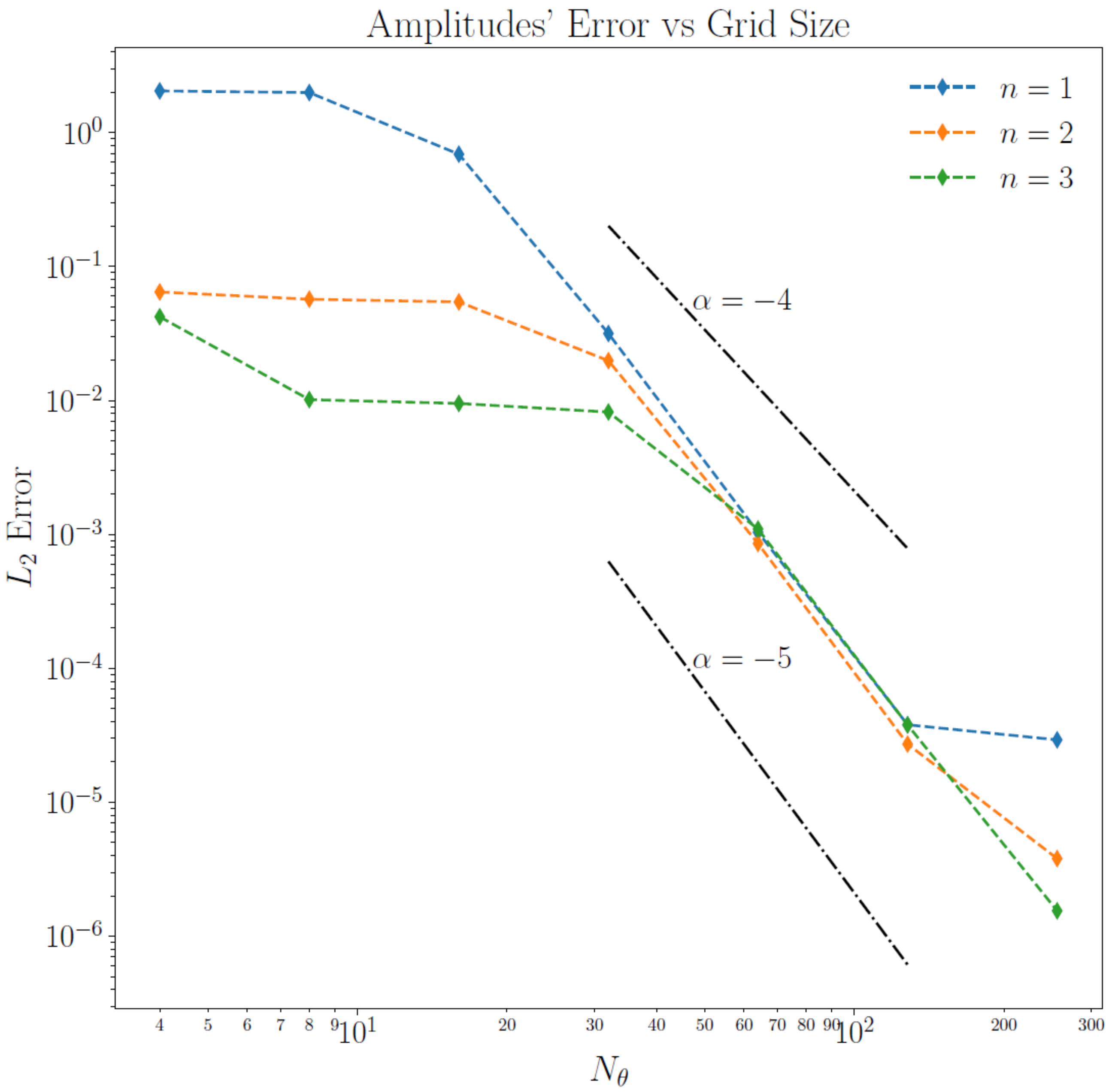
The dissipation induces higher modes to decay faster



$1{\times}512$ grid,
$\delta t = 1{\times}10^{-4}$, $\tau = 2{\times}10^{-3}$, $R = 1$ (WENO-5)

Sound wave convergence

Again, near fifth-order convergence for all the modes considered

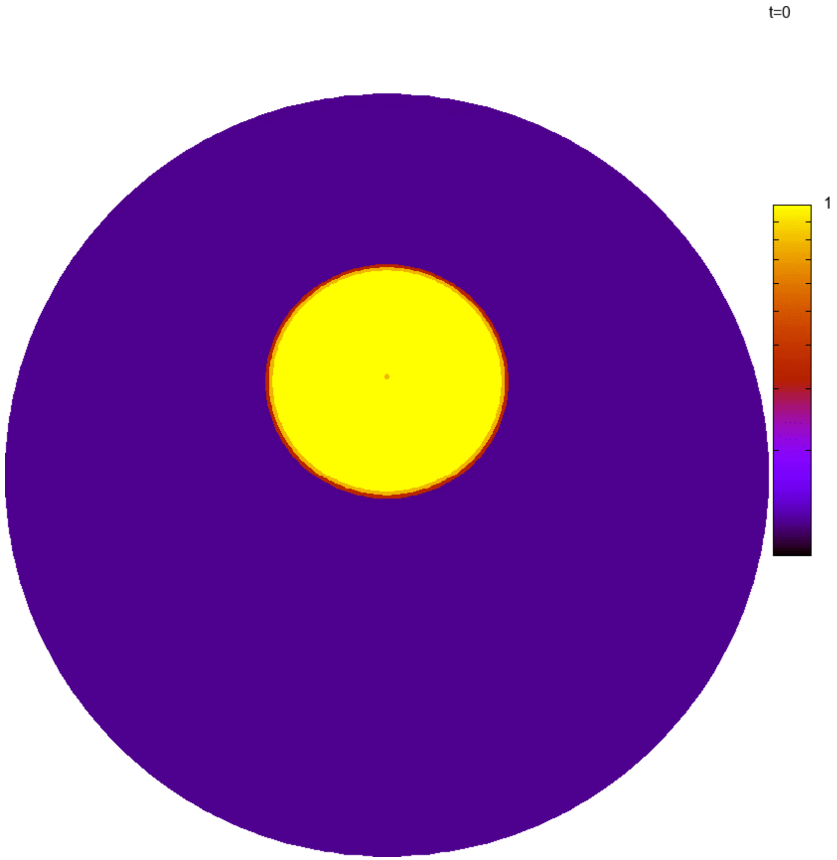


$1 \times N_\theta$ grids, D2Q16, $\delta t = 2 \times 10^{-5}$, $\tau = 2 \times 10^{-5}$, $R = 1$ (WENO-5)

Bonus Flows

Shock waves

The high-order scheme employed allows to reproduce highly compressible fluids



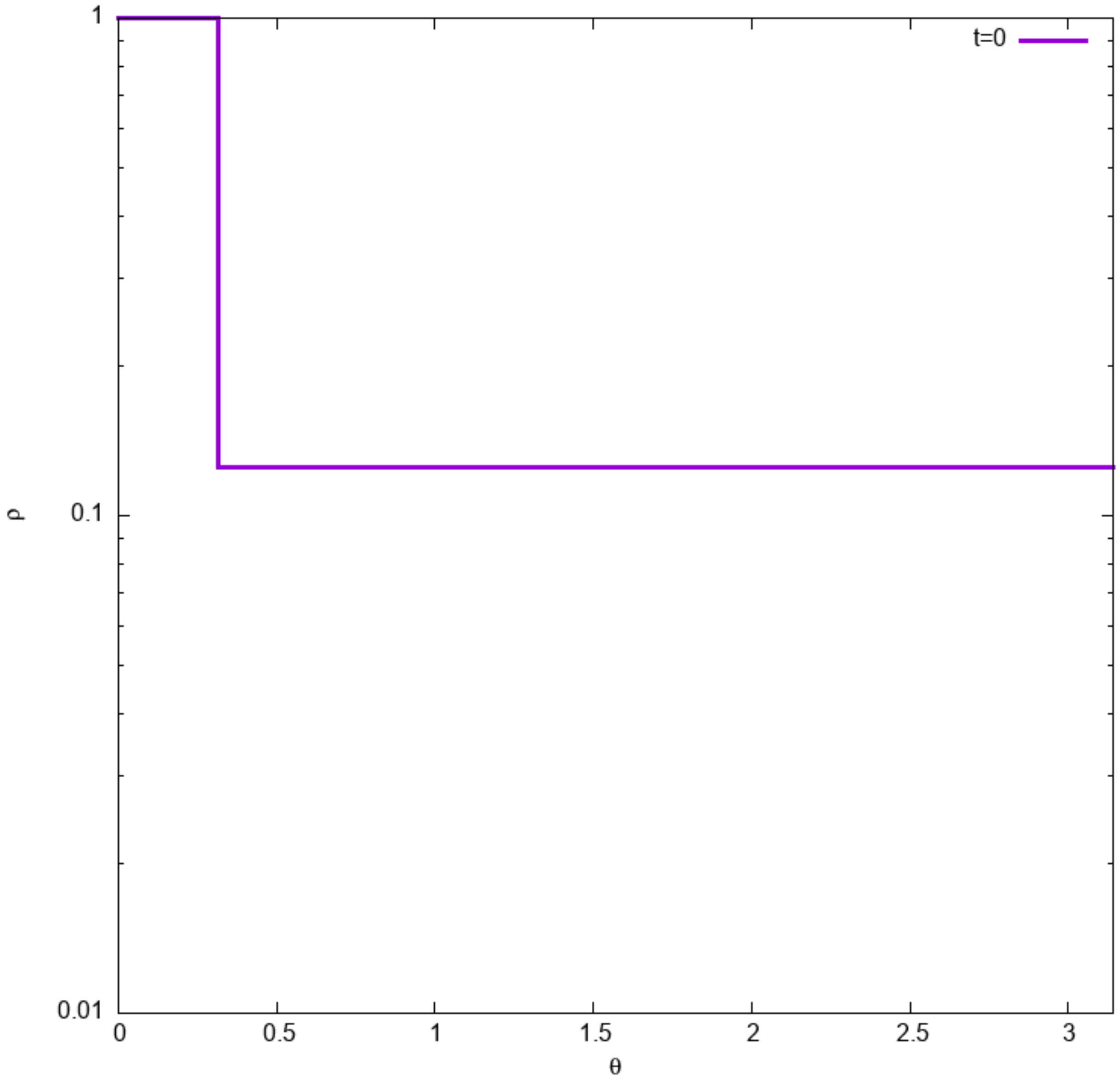
Density profile of an initial shock wave

Riemann problem

We take the initial density profile:

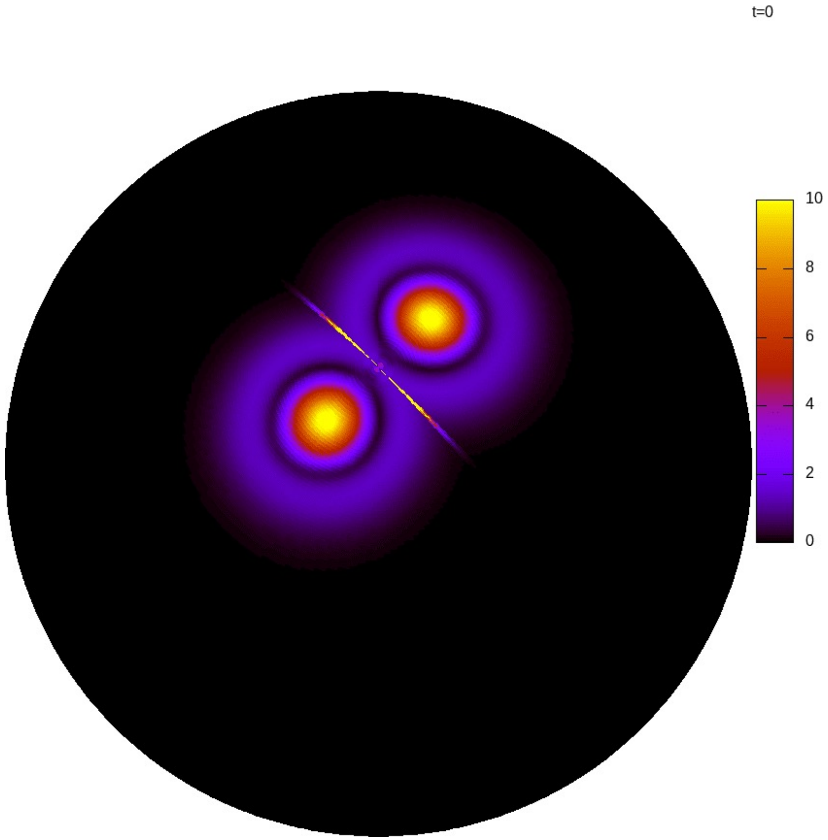
$$\rho(\theta) = \begin{cases} \rho_1, & \theta < \theta_{\rm disc}, \\ \rho_2, & \theta > \theta_{\rm disc}, \end{cases}$$

With: $u^{\hat{\varphi}} = u^{\hat{\theta}} = 0$



2D Flows

We reproduce the dynamics of two vortexes with initial velocity profile given by:



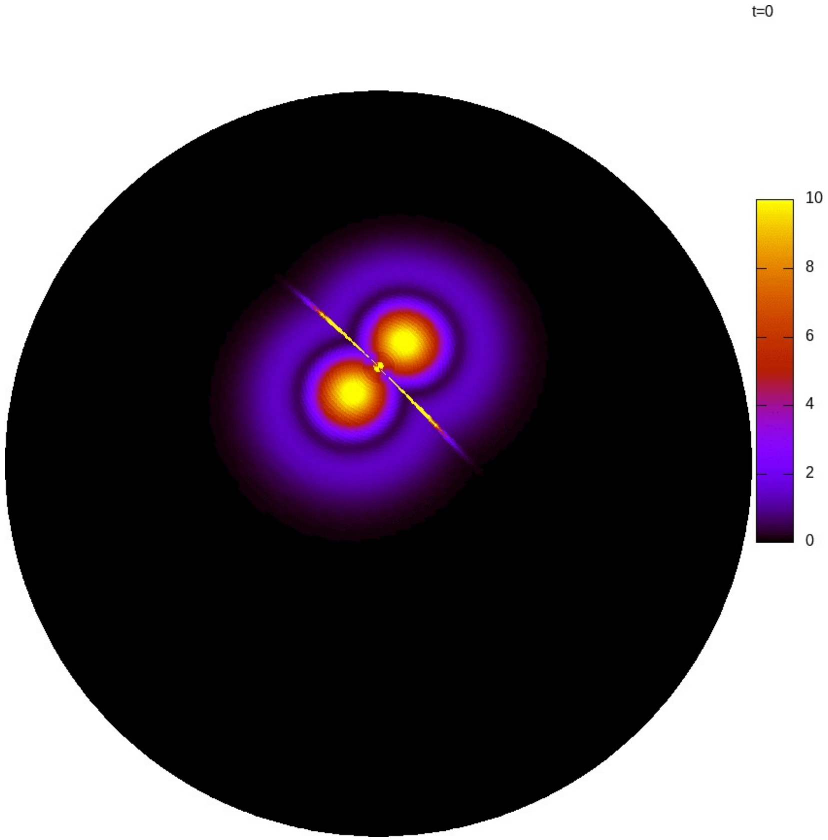
$c_1 = 0.8, c_2 = 1.2, Re = 3\times10^4$

$$u = \begin{cases} 4(y-1)e^{0.3\left(1-l_1^2\right)} & \text{if } x<1, z>1, \\ 4(y-1)e^{0.3\left(1-l_2^2\right)} & \text{if } x>1, z>1, \\ 0 & \text{otherwise,} \end{cases}$$

$$v = \begin{cases} -4\,(x-c_1)\,e^{0.3\left(1-l_1^2\right)} & \text{if } x<1, z>1, \\ -4\,(x-c_2)\,e^{0.3\left(1-l_2^2\right)} & \text{if } x>1, z>1, \\ 0 & \text{otherwise,} \end{cases}$$

$$w = 0$$

$$l_{1,2} = \sqrt{\left(x-c_{1,2}\right)^2 + (y-1)^2}/0.08$$



$c_1 = 0.9, c_2 = 1.1, Re = 3\times10^4$

Conclusions and outlook

Conclusions

- Extension of LBM to spherical surface using vielbein formalism

- New solutions for benchmark problems:
  - Shear wave
  - Sound wave

- Fully compressible solver (Riemann problem on the sphere)

- 2D flows vortex dynamics on the sphere

- Finite difference → high-order scheme

Thank you for your attention

Backup

Jacobi polynomials

The Jacobi polynomials are solutions to the following differential eq:

$$\left[(1-x^2)\frac{d^2}{dx^2} + [\beta - \alpha - (\alpha+\beta+2)x]\frac{d}{dx} + n(n+\alpha+\beta+1)\right] P_n^{(\alpha,\beta)}(x) = 0$$

They can be represented in series form as:

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{x-1}{2}\right)^m$$

Shear wave:

$$F_n(\theta) = \sqrt{\frac{(2n+1)(4n+3)}{4(n+1)}} P_n^{(1,-\frac{1}{2})}(\cos 2\theta)$$

Sound wave:

$$F_n(\theta) = (-1)^n \sqrt{\frac{n(4n-1)}{2n-1)}} P_n^{(-1,-\frac{1}{2})}(\cos 2\theta)$$

Vorticity

We define vorticity on the spherical surface as:

$$\omega = \frac{\mathbf{e}_r}{r\sin\theta}\left(\frac{\partial(u_\varphi \sin\theta)}{\partial\theta} - \frac{\partial u_\theta}{\partial\varphi}\right) + \frac{\mathbf{e}_\theta}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(r u_\varphi)}{\partial r}\right) + \frac{\mathbf{e}_\varphi}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right)$$

WENO-5

Weighted Essentially Non-Oscillatory algorithm for $v_{s+1/2} > 0$:

$$\mathcal{F}_{s+1/2} = \bar{\omega}_1 \mathcal{F}^1_{s+1/2} H + \bar{\omega}_2 \mathcal{F}^2_{s+1/2} + \bar{\omega}_3 \mathcal{F}^3_{s+1/2}$$

$$\mathcal{F}^1_{s+1/2} = \frac{1}{3} J_{s-2} - \frac{7}{6} J_{s-1} + \frac{11}{6} J_s$$

$$\mathcal{F}^2_{s+1/2} = -\frac{1}{6} J_{s-1} + \frac{5}{6} J_s + \frac{1}{3} J_{s+1}$$

$$\mathcal{F}^3_{s+1/2} = \frac{1}{3} J_s + \frac{5}{6} J_{s+1} - \frac{1}{6} J_{s+2}$$

$$\bar{\omega}_r = \frac{\widetilde{\omega}_r}{\widetilde{\omega}_1 + \widetilde{\omega}_2 + \widetilde{\omega}_3}, \quad \widetilde{\omega}_r = \frac{\delta_r}{\sigma_r^2}$$

$$\sigma_1 = \frac{13}{12} \left(J_{s-2} - 2J_{s-1} + J_s\right)^2 + \frac{1}{4} \left(J_{s-2} - 4J_{s-1} + 3J_s\right)^2,$$

$$\sigma_2 = \frac{13}{12} \left(J_{s-1} - 2J_s + J_{s+1}\right)^2 + \frac{1}{4} \left(J_{s-1} - J_{s+1}\right)^2,$$

$$\sigma_3 = \frac{13}{12} \left(J_s - 2J_{s+1} + J_{s+2}\right)^2 + \frac{1}{4} \left(3J_s - 4J_{s+1} + J_{s+2}\right)^2.$$

$$\delta_1 = 1/10, \quad \delta_2 = 6/10, \quad \delta_3 = 3/10$$

Discrete Boltzmann Eq on the sphere

$$\frac{\partial f_{\mathbf{k};s,q}}{\partial t} + \overbrace{\frac{1}{R\sin\theta_s}\left[\frac{\mathcal{F}^{\theta}_{s+\frac{1}{2},q} - \mathcal{F}^{\theta}_{s-\frac{1}{2},q}}{\delta\theta} + \frac{\mathcal{F}^{\varphi}_{s,q+\frac{1}{2}} - \mathcal{F}^{\varphi}_{s,q-\frac{1}{2}}}{\delta\varphi}\right]}^{\text{Advection}}$$

$$+ \underbrace{\frac{\cos\theta_s}{R\sin\theta_s}\left[v_{k_\varphi}^2\left(\frac{\partial f_{\mathbf{k};s,q}}{\partial v_{k_\theta}}\right) - v_{k_\theta}\left(\frac{\partial(f_{\mathbf{k};s,q}v_{k_\varphi})}{\partial v_{k_\varphi}}\right)\right]}_{\text{Internal forcing}} = \underbrace{-\frac{1}{\tau}\left[f_{\mathbf{k};s,q} - f^{\text{eq}}_{\mathbf{k};s,q}\right]}_{\text{Collisions}}$$

$$\theta_s = \frac{\pi}{N_\theta}(s - 1/2) \qquad (1 \leq s \leq N_\theta)$$

$$\varphi_q = \frac{2\pi}{N_\varphi}(q - 1/2) \qquad (1 \leq q \leq N_\varphi).$$

Komissarov scheme

The advection along θ is treated with the following scheme:

$$\left[\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(f_{\mathbf{k}}v_{k_\theta}\sin\theta)\right]_{s,q} \simeq \frac{1}{R}\left[\frac{\mathcal{F}^{\theta}_{s+\frac{1}{2},q}\sin\theta_{s+\frac{1}{2}} - \mathcal{F}^{\theta}_{s-\frac{1}{2},q}\sin\theta_{s-\frac{1}{2}}}{\cos\theta_{s-\frac{1}{2}} - \cos\theta_{s+\frac{1}{2}}}\right]$$

As opposed to:

$$\left[\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(f_{\mathbf{k}}v_{k_\theta}\sin\theta)\right]_{s,q} \simeq \frac{1}{R\sin\theta_s}\left[\frac{\mathcal{F}^{\theta}_{s+\frac{1}{2},q} - \mathcal{F}^{\theta}_{s-\frac{1}{2},q}}{\delta\theta}\right]$$

Equilibrium Distribution Function

Maxwell-Boltzmann distribution function:

$$f^{\mathrm{eq}}(\mathbf{x}, \mathbf{u}, t, T, \mathbf{v}) = \frac{\rho}{(2\pi RT)^{\frac{D}{2}}} \exp\left[-\frac{(\mathbf{v}-\mathbf{u})^2}{2RT}\right]$$

We treated only isothermal flows

$$\nu = \frac{\eta}{\rho} = \frac{\zeta}{\rho} = \frac{\tau k_{\mathrm{B}} T}{m}$$

Equilibrium expansion in D2Q16:

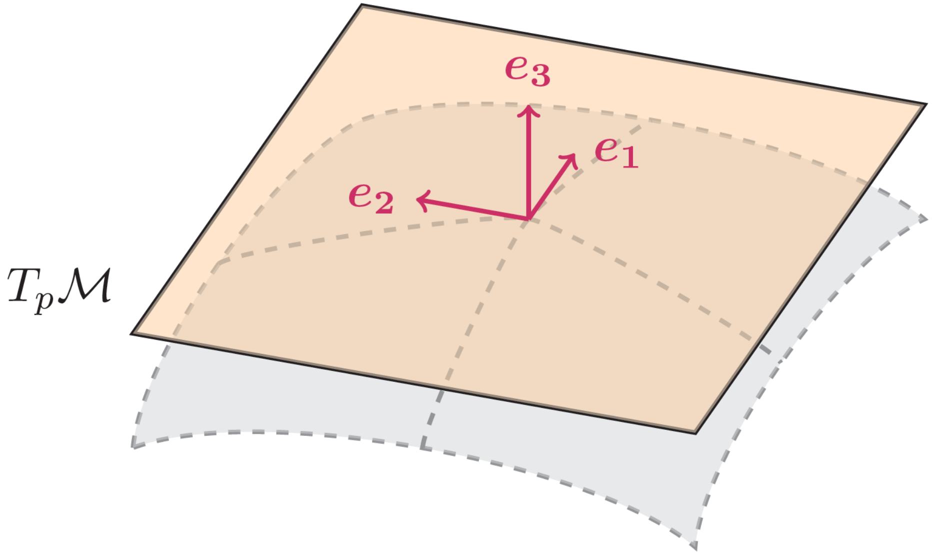
$$f_{\mathbf{k}}^{Q=4,\mathrm{eq}} = n w_{\mathbf{k}} \left[1 + \mathbf{v_k}\cdot\mathbf{u} + \frac{1}{2}\left((\mathbf{v_k}\cdot\mathbf{u})^2 - u^2\right) + \frac{1}{6}\mathbf{v_k}\cdot\mathbf{u}\left((\mathbf{v_k}\cdot\mathbf{u})^2 - 3u^2\right)\right]$$

Vielbein formalism

The Cartan coefficients and spin connections are defined as:

$$c_{\hat{a}\hat{b}}{}^{\hat{c}} := \langle [\boldsymbol{e}_{\hat{a}}, \boldsymbol{e}_{\hat{b}}], \boldsymbol{\omega}^{\hat{c}} \rangle = \omega^{\hat{c}}_{c}(e^{a}_{\hat{a}}\partial_a e^{c}_{\hat{b}} - e^{b}_{\hat{b}}\partial_b e^{c}_{\hat{a}})$$

$$\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} = \frac{1}{2}\delta^{\hat{a}\hat{d}}(c_{\hat{d}\hat{b}\hat{c}} + c_{\hat{d}\hat{c}\hat{b}} - c_{\hat{b}\hat{c}\hat{d}})$$



Continuity and NSE on the sphere

- Continuity equation:

$$\partial_t \rho + \frac{1}{R\sin\theta}\partial_\theta\left(\rho\sin\theta u^{\hat\theta}\right) + \frac{1}{R\sin\theta}\partial_\varphi\left(\rho u^{\hat\varphi}\right) = 0$$

- Navier-Stokes equations:

$$\rho\left[\partial_t u^{\hat\theta} + \frac{1}{R}u^{\hat\theta}\partial_\theta u^{\hat\theta} + \frac{1}{R\sin\theta}u^{\hat\varphi}\partial_\varphi u^{\hat\theta} - \frac{\cos\theta}{R\sin\theta}u^{\hat\varphi}u^{\hat\varphi}\right] =$$
$$- \frac{1}{R}\partial_\theta p_{\mathrm{i}} + \eta\left[\Delta u^{\hat\theta} - 2\frac{\cos\theta}{R^2\sin^2\theta}\partial_\varphi u^{\hat\varphi} - \frac{1}{R^2\sin^2\theta}u^{\hat\theta}\right]$$

Non-dimensionalization

All physical quantities presented are non-dimensionalized w.r.t. background fluid parameters:

$$\rho_0 = T_0 = P_0 = 1$$

The velocities are given w.r.t. reference velocity:

$$c_0 = \sqrt{P_0/\rho_0} = 1$$

Times are given in lattice units:

$$t = n\,\delta t$$