Permutation Group $S(N)$ and Young diagrams

$S(N)$: order = $N!$ huge representations but allows general analysis, with many applications. Example

$S(3) = C_{3v}$

In $C_{3v}$ reflections $\rightarrow$ transpositions.

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$C_3$</th>
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<td>$C_3$</td>
<td>$C_3^2$</td>
<td>$E$</td>
<td></td>
</tr>
</tbody>
</table>

$C_{3v}$: All operators = reflections or products of reflections

$S(3)$: All operators = permutations = exchanges or products of transpositions
They are isomorphous

Permutations of Group elements are the basis of the regular representation of any Group.
Permutation Group $S(N)$ and Young diagrams

$S(N)$ : order= $N!$ huge representations but allows general analysis, with many applications.

Young diagrams are in one-to one correspondence with the irreps of $S(N)$

Rule: partition $N$ in not increasing integers: e.g.

$$8 = 3 + 2 + 2 + 1$$

Draw a diagram with 3 boxes, and below two boxes twice and finally one box, all lined up to the left

This corresponds to an irrep of $S(8)$
Young Diagrams for $S(3) = C_{3v}$ and partitions of 3 in not increasing integers
(lower rows cannot be longer)

Each Young Diagram for $S(N)$ corresponds to an irrep

Diagrams that are obtained from each other by interchanging rows and columns are conjugate diagrams. The representations are said conjugate, like these.

\begin{align*}
3 & \quad \begin{array}{ccc}
\text{\hspace{1cm}} \\
\text{\hspace{1cm}} \\
\text{\hspace{1cm}} \\
\end{array} & \quad \begin{array}{c}
\text{\hspace{1cm}} \\
\text{\hspace{1cm}} \\
\end{array} \\
2+1 & \quad \begin{array}{ccc}
\text{\hspace{1cm}} \\
\text{\hspace{1cm}} \\
\text{\hspace{1cm}} \\
\end{array} & \quad \begin{array}{c}
\text{\hspace{1cm}} \\
\text{\hspace{1cm}} \\
\end{array} & \quad 1+1+1
\end{align*}
**Young Diagrams for \( S(3) = C_{3v} \) and correspondence to irreps**

<table>
<thead>
<tr>
<th>( C_{3v} )</th>
<th>( I )</th>
<th>( 2C_3 )</th>
<th>( 3\sigma_v )</th>
<th>( g = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>symmetric</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>antisymmetric</td>
</tr>
<tr>
<td>( E )</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>mixed</td>
</tr>
</tbody>
</table>

Theorem: if \( P \in S(N) \) (is a permutation) and \( \alpha \) and \( \beta \) conjugate irreps of \( S(N) \),

\[
D_{jk}^{(\alpha)}(P) = (-)^P D_{kj}^{(\beta)}(P^{-1})
\]
Young Tableaux (Tables)

The Young tables or Young tableaux for S(N) are obtained from the Young diagrams by inserting numbers from 1 to N so that they grow along every line and every column.
Young Projectors

The Young tables or Young tableaux are associated to symmetrization along lines and antisymmetrization along columns. In this way one projects onto irreps

\[
\begin{bmatrix}
1 & 2 & 3 \\
\end{bmatrix} \quad \rightarrow \quad S_{123} = \text{symmetrizer}
\]

\[
S_{123} f(1,2,3) = [1 + P_{12} + P_{13} + P_{23} + P_{12}P_{13} + P_{13}P_{12}] f(1,2,3)
\]

\[
\begin{bmatrix}
1 & 3 \\
2
\end{bmatrix} \quad \rightarrow \quad A_{12} S_{13} \neq S_{13} A_{12}
\]

\[
A_{12} S_{13} f(1,2,3) = A_{12} [f(1,2,3) + f(3,2,1)] = f(1,2,3) + f(3,2,1) - f(2,1,3) - f(3,1,2)
\]

\[
\begin{bmatrix}
1 & 2 \\
3
\end{bmatrix} \quad \rightarrow \quad A_{13} S_{12}
\]

\[
A_{13} S_{12} f(1,2,3) = A_{13} [f(1,2,3) + f(2,1,3)] = f(1,2,3) + f(2,1,3) - f(3,2,1) - f(2,3,1)
\]
Rule: first, symmetrize.

There are two tables with mixed permutation symmetry (i.e. not fully symmetric or antisymmetric) due to degeneracy 2 of the irrep E. One can show that this is general. In the Young tables for $S(N)$, the m-dimensional irreps occur in m different tableaux.

Antisummetrizer

\[
\begin{array}{ccc}
1 \\
2 \\
3
\end{array} \rightarrow A_{123}
\]

\[
A_{123}f(1,2,3) = (1 - P_{12})(1 - P_{13})(1 - P_{23})f(1,2,3)
= [1 - P_{12} - P_{13} - P_{23} + P_{12}P_{13} + P_{13}P_{12}]f(1,2,3)
\]

Counting the number of diagrams can be long, but there is a shortcut
Hook - Capitan Uncino

Film (1991)

6,8/10 · IMDb
3,5/5 · MYmovies.it
30% · Rotten Tomatoes


Prima data di uscita: 8 dicembre 1991 (CA)
Regista: Steven Spielberg
Durata: 144 minuti
Musica composta da: John Williams
Sceneggiatura: Nick Castle, James V. Hart, Malia Scotch Marmo

Cast

Robin Williams
Dustin Hoffman
Amber Scott
Julia Roberts
Bob Hoskins

Peter Pan
Capitan Uncino
Maggie Banning
Campanellino
Sougna
Dimension of a representation and hook-length formula:
An example for $S_\text{N}$ with $N=13$

Hook length of a box = 1 + number of boxes on the same line on the right of it + number of boxes in the same column below it

$$\Pi = \text{product of hook lengths} = 9.7.5.4.2.6.4.2.3 = 362880$$
(red from first line, blue from second line)

$n=$number of boxes =13

dimension=number of tableaux of this shape = $\frac{n!}{\Pi} = 17160$
**Projection operators: verification for S(3)**

\[ S(3) = C_{3v} \]

All operators = reflections or products of reflections

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<td>1</td>
<td>( z )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>( R_z )</td>
</tr>
<tr>
<td>( E )</td>
<td>2</td>
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<td>0</td>
<td>( (x, y) )</td>
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<td>( E )</td>
</tr>
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</table>

irrep \( E \) of \( C_{3v} \):

\[
D(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(C_3) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad D(C_3^2) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}
\]

\[
D(\sigma_a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(\sigma_c) = \begin{pmatrix} -c & -s \\ -s & c \end{pmatrix}, \quad D(\sigma_b) = \begin{pmatrix} -c & s \\ s & c \end{pmatrix}
\]

\[
c = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \quad s = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}
\]
Projection on component $x$ of $E$

\[ P_{xx}^E = \frac{2}{6} \sum_R D_{xx}^E(R)^* R = 1 - \frac{1}{2} \left( C_3 + C_3^2 \right) - \sigma_a + \frac{1}{2} \left( \sigma_b + \sigma_c \right) \]

Now write rotations in terms of reflections

$\sigma_a$ exchanges $b$ and $c$; reflections are exchanges.

\[ C_3 = \sigma_a \sigma_b \quad C_3^2 = \sigma_a \sigma_c \]

\[ P_{xx}^E = 1 - \frac{\sigma_a}{2} \left( \sigma_b + \sigma_c \right) - \sigma_a + \frac{1}{2} \left( \sigma_b + \sigma_c \right) \]

\[ = (1 - \sigma_a) \left( 1 + \frac{\sigma_b + \sigma_c}{2} \right) \]

In $C_{3v}$ reflections $\rightarrow$ transpositions, i.e., exchanges.
\[ P^E_{xx} = 1 - \frac{\sigma_a}{2} \left( \sigma_b + \sigma_c \right) - \sigma_a + \frac{1}{2} \left( \sigma_b + \sigma_c \right) \]

\[ = (1 - \sigma_a)(1 + \frac{\left( \sigma_b + \sigma_c \right)}{2}) \]

Now we can introduce symmetrizer and antisymmetrizer

\[ S(b, c) \equiv S(2, 3) = \frac{1 + \sigma_a}{2} = \frac{1 + P_{23}}{2} \]
\[ A(b, c) \equiv A(2, 3) = \frac{1 - \sigma_a}{2} = \frac{1 - P_{23}}{2} \]

\[ P^E_{xx} = A_{23}(S_{12} + S_{13}) \]

Recall: the permutations of N objects are the basis of the Regular Representation of S(N)

In S(N), transpositions are the basis of the regular representation. So we are projecting from the regular representation to irreps of S(3).

Similar rules apply for S(N)
Young Tableaux for $S(4)$

Conjugate representations (conjugate diagrams) are obtained from each other by exchanging rows with columns.

The number of tableaux for each diagram is the degeneracy of the irrep.
Example: projection onto

\[
P(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}) = A_{24} A_{13} S_{34} S_{12}
\]

\[
A_{24} A_{13} S_{34} S_{12} \alpha(1) \beta(2) \gamma(3) \delta(4) = A_{24} A_{13} \left[ \alpha(1) \beta(2) + \beta(1) \alpha(2) \right] \left[ \gamma(3) \delta(4) + \delta(3) \gamma(4) \right]
\]

Antisymmetrize on 1 and 3 and get

\[
= A_{24} \left[ \alpha(1) \beta(2) + \beta(1) \alpha(2) \right] \left[ \gamma(3) \delta(4) + \delta(3) \gamma(4) \right] - \left[ \alpha(3) \beta(2) + \beta(3) \alpha(2) \right] \left[ \gamma(1) \delta(4) + \delta(1) \gamma(4) \right]
\]

Antisymmetrize on 2 and 4 and get the final result:

\[
= \left[ \alpha(1) \beta(2) + \beta(1) \alpha(2) \right] \left[ \gamma(3) \delta(4) + \delta(3) \gamma(4) \right] - \left[ \alpha(3) \beta(2) + \beta(3) \alpha(2) \right] \left[ \gamma(1) \delta(4) + \delta(1) \gamma(4) \right] - \\
\left[ \alpha(1) \beta(4) + \beta(1) \alpha(4) \right] \left[ \gamma(3) \delta(2) + \delta(3) \gamma(2) \right] + \left[ \alpha(3) \beta(4) + \beta(3) \alpha(4) \right] \left[ \gamma(1) \delta(2) + \delta(1) \gamma(2) \right]
\]
Young tableaux and spin eigenfunctions

Consider the eigenstates \( |S, M_s > \) obtained by solving the eigenvalue problems for \( S^2 \) and \( S_z \). Several eigenstates of \( S^2 \) and \( S_z \) with the same quantum numbers can occur.

Example: \( N = 3 \) electron spins

\[ 2^3 = 8 \] states, maximum spin = \( 3/2 \) \( \rightarrow \) 4 states.

Hilbert space: 1 quartet and 2 doublets.

Any permutation of the spins sends an \( |S, M_s > \) eigenfunction into a linear combination of the eigenfunctions with the same eigenvalues \( S, M_s \); in other terms, the \( S, M_s \) quantum numbers label subspaces of functions that do not mix under permutations.
The reason is that \( \vec{S} = \sum_i \vec{S}_i \) is invariant under permutations of \( \vec{S}_i \).

*Within each permutation symmetry subspace, by a technique based on shift operators we shall learn to produce \( S \) and \( M_s \) eigenfunctions that besides bearing the spin labels also form a basis of irreps of \( S(N) \).*

*We can use the example of \( N=3 \) electron spins, \( 2^3=8 \) states, maximum spin = \( 3/2 \) \( \rightarrow \) 4 states. Hilbert space: 1 quartet and 2 doublets.*
With $M_s = 3/2$

the only state is quartet $|↑↑↑>\rangle$, which is invariant for any permutation.

Acting on $|↑↑↑>\rangle$ with $S^-$

get $|3/2, 1/2\rangle = \frac{1}{\sqrt{3}} (|↑↑↓> + |↑↓↑> + |↓↑↑> )$.

This is invariant for spin permutations, too, and belongs to the $A_1$ irrep of $S(3)$.

The (total-symmetric) shift operators preserve the permutation symmetry, and all the $2M_s+1$ states belong to the same irrep.

Acting again with $S^-$ we get

$|3/2, -1/2\rangle = \frac{1}{\sqrt{3}} (|↑↓↓> + |↓↑↓> + |↓↓↑> )$

$|3/2, -3/2\rangle = |↓↓↓> \rangle$
The quartenet $|\frac{3}{2}, \frac{1}{2} \rangle = \frac{1}{\sqrt{3}} (|\uparrow\uparrow\downarrow \rangle + |\uparrow\downarrow\uparrow \rangle + |\downarrow\uparrow\uparrow \rangle)$ involves $|\uparrow\uparrow\downarrow \rangle, |\uparrow\downarrow\uparrow \rangle, |\downarrow\uparrow\uparrow \rangle$ according to $A_1$ of $S(3)$. Out of these we can also build a 3d subspace with $M_s = 1/2$.

A two-d subspace is orthogonal to $|\frac{3}{2}, \frac{1}{2} \rangle$.

The ortogonal subspace involving one down spin yields two different doublets with $M_s = 1/2$

$$|\frac{1}{2}, \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} [ |\uparrow\uparrow\downarrow \rangle - |\uparrow\downarrow\uparrow \rangle], \quad |\frac{1}{2}, \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} [ |\downarrow\uparrow\uparrow \rangle - |\uparrow\downarrow\uparrow \rangle]$$

We can orthonormalize the doublets:

$$|\frac{1}{2}, \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} [ |\uparrow\uparrow\downarrow \rangle - |\uparrow\downarrow\uparrow \rangle], \quad |\frac{1}{2}, \frac{1}{2} \rangle = \frac{1}{\sqrt{6}} [ 2 |\downarrow\uparrow\uparrow \rangle - |\uparrow\downarrow\uparrow \rangle - |\uparrow\uparrow\downarrow \rangle]$$

Why two? What good quantum number distinguishes these two states? It is the permutation symmetry, which admits a degenerate representation.
Looking at the doublets:

\[ \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}} \left[ \left| \uparrow \downarrow \downarrow \right\rangle - \left| \uparrow \uparrow \uparrow \right\rangle \right], \quad \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{1}{\sqrt{6}} \left[ \left| \downarrow \uparrow \uparrow \right\rangle - \left| \uparrow \downarrow \uparrow \right\rangle - \left| \uparrow \uparrow \downarrow \right\rangle \right] \]

we recognize irrep \( E \) of \( C_{3v} \) (\( x \) odd for \( 2 \leftrightarrow 3 \))

\[ |E, x\rangle = \frac{1}{\sqrt{2}} \left[ |\alpha \alpha \beta\rangle - |\alpha \beta \alpha\rangle \right], \quad |E, y\rangle = \frac{1}{\sqrt{6}} \left[ 2 |\beta \alpha \alpha\rangle - |\alpha \beta \alpha\rangle - |\alpha \alpha \beta\rangle \right] \]

within subspace with \( Ms = 1/2 \)

Moreover, by the spin shift operators each yields its \( |1/2, -1/2\rangle \) companion.

A quarted and 2 doublets exhaust all the \( 2^3 \) states available for \( N=3 \), and there is no space for the \( A_2 \) irrep.

This is general:

since spin \( 1/2 \) has two states available, any spin wave function belongs to a Young diagram with 1 or 2 rows.
Conclusion: a system consisting of $N$ spins $1/2$.

The set of spin configurations, like $\alpha\alpha\beta\alpha\alpha\ldots\alpha\beta$ can be used to build a representation of the permutation Group $S(N)$.

One can build spin eigenstates by selecting the number of up and down spins according to $M_s$ and then projecting with the Young tableaux.
For the CuO$_4$ model cluster

(1) Find the irreps of the one-electron orbitals.

(2) Consider this cluster with 4 fermions, in the $S_z = 0$ sector. Classify the 4-body states with the irreps of the Group.

Consider the Group operators acting on the basis of atomic orbitals (1,2,3,4,5). Atoms that do not move contribute 1 to the character. The characters of the representation $\Gamma(1)$ with one electron are

$\chi(E) = 5$, $\chi(C_2) = 1$, $\chi(2C_4) = 1$, $\chi(2\sigma_v) = 3$, $\chi(2\sigma_d) = 1$. Applying the LOT one finds $\Gamma(1) = 2A_1 + E + B_1$. 

<table>
<thead>
<tr>
<th>$C_{4v}$</th>
<th>$I$</th>
<th>$C_2$</th>
<th>$2C_4$</th>
<th>$2\sigma_v$</th>
<th>$2\sigma_d$</th>
<th>$g = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$z$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>$R_z$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>$x^2 - y^2$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>$xy$</td>
</tr>
<tr>
<td>$E$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$(x, y)$</td>
</tr>
</tbody>
</table>
The $CuO_4$ Model Cluster

4 Fermion case

Basis: $(i,j,k,l) \equiv \{i \uparrow, j \uparrow, k \downarrow, l \downarrow\}$

\[
\begin{pmatrix} 5 \choose 2 \end{pmatrix} \text{ choices of } i,j \quad \begin{pmatrix} 5 \choose 2 \end{pmatrix} \text{ choices of } k,l \quad \Rightarrow \begin{pmatrix} 5 \choose 2 \end{pmatrix}^2 = 100 \quad \text{configurations}
\]

Invariant configurations:

C2: $(4, 2, 4, 2), (4, 2, 5, 3), (5, 3, 4, 2), (5, 3, 5, 3)$ are invariant, +1 to character each

C4: none is invariant
\( \sigma_x : (2, 1, 2, 1), (2, 1, 4, 1), (2, 1, 4, 2), (4, 1, 2, 1), (4, 1, 4, 1), (4, 1, 4, 2), (4, 2, 2, 1), (4, 2, 4, 1), (4, 2, 4, 2), (5, 3, 5, 3) \) invariant, +1 each

\( (2, 1, 5, 3) \rightarrow (2, 1, 3, 5) = -(2, 1, 5, 3) \)

change sign since order requires exchange of creation operators; also,

\( (4, 1, 5, 3), (4, 2, 5, 3), (5, 3, 2, 1), (5, 3, 4, 1), (5, 3, 4, 2) \) -1 each
\( \sigma_{ne} : (5, 2, 5, 2), (5, 2, 4, 3), (4, 3, 5, 2), (4, 3, 4, 3) \) invariant

\[
\begin{array}{c|cccccc|c}
C_{4v} & I & C_2 & 2C_4 & 2\sigma_v & 2\sigma_d & g = 8 \\
\hline
A_1 & 1 & 1 & 1 & 1 & 1 & z \\
A_2 & 1 & 1 & 1 & -1 & -1 & R_z \\
B_1 & 1 & 1 & -1 & 1 & -1 & x^2 - y^2 \\
B_2 & 1 & 1 & -1 & -1 & 1 & xy \\
E & 2 & -1 & 0 & 0 & 0 & (x, y) \\
\end{array}
\]

\[ n_i = \frac{1}{N_G} \sum_{R \in G} \chi(R) \chi^{(i)}(R)^* \]

\[ \Gamma(4) = 15A_1 \oplus 11A_2 \oplus 24E \oplus 13B_1 \oplus 13B_2 \]
The $Cu_5O_4$ Model Cluster

Classify the 4-holes states in the $S_z=0$ sector by the $C_{4v}$ irreps

Basis: $(i,j,k,l)= l i+j+k-m-l$

\[
\binom{9}{2}^2 = 1296 \text{ configurations}
\]

\[
\Gamma(4) = 184A_1 \oplus 144A_2 \oplus 320E \oplus 176B_1 \oplus 152B_2
\]
Let \( \phi_i^{(\alpha)}(x_1, \ldots, x_N) \) be \( N \) electron amplitude depending on space coordinates only.

We take \( \phi_i^{(\alpha)}(x_1, \ldots, x_N) \) as component \( i \) of irrep \( \alpha \) of \( S_N \), \( m \) times degenerate.

Then if \( P = \) permutation \( \in S_N \)

\[
P\phi_i^{(\alpha)}(x_1, \ldots, x_N) = \sum_j m \phi_j^{(\alpha)}(x_1, \ldots, x_N) D_{ji}^{(\alpha)}(P).
\]

Now let \( \chi_q^{(\beta)}(\sigma_1, \ldots, \sigma_N) \) be \( N \) electron amplitude depending on spin coordinates only.

We take \( \chi_q^{(\beta)}(\sigma_1, \ldots, \sigma_N) \) as component \( q \) of irrep \( \beta \) of \( S_N \), \( m \) times degenerate.

Then if \( P = \) permutation \( \in S_N \)

\[
P\chi_q^{(\beta)}(\sigma_1, \ldots, \sigma_N) = \sum_n m \chi_n^{(\beta)}(\sigma_1, \ldots, \sigma_N) D_{nq}^{(\beta)}(P).
\]

The full electron wave function must be of the form:

\[
\Psi = \sum_k^m \phi_k^{(\alpha)}(x_1, \ldots, x_N) \chi_k^{(\beta)}(\sigma_1, \ldots, \sigma_N), \text{ with } \alpha \text{ and } \beta \text{ same degeneracy, such that}
\]

\[
P\Psi = (-1)^P \Psi.
\]
Putting all together:

\[
P\phi_i^{(\alpha)}(x_1, \ldots, x_N) = \sum_j m_j \phi_j^{(\alpha)}(x_1, \ldots, x_N) D_{ji}^{(\alpha)}(P)
\]

\[
P\chi_q^{(\beta)}(\sigma_1, \ldots, \sigma_N) = \sum_n m_n \chi_n^{(\beta)}(\sigma_1, \ldots, \sigma_N) D_{nq}^{(\beta)}(P)
\]

\[
\Psi = \sum_k m_k \phi_k^{(\alpha)}(x_1, \ldots, x_N) \chi_k^{(\beta)}(\sigma_1, \ldots, \sigma_N), \alpha \text{ and } \beta \text{ same degeneracy}
\]

with \( P\Psi = \sum_k m_k [P\phi_k^{(\alpha)}(x_1, \ldots, x_N)][P\chi_k^{(\beta)}(\sigma_1, \ldots, \sigma_N)] \)

\[
= \sum_k \sum_j m_j \phi_j^{(\alpha)}(x_1, \ldots, x_N) D_{jk}^{(\alpha)}(P) \sum_n m_n \chi_n^{(\beta)}(\sigma_1, \ldots, \sigma_N) D_{nk}^{(\beta)}(P)
\]

\[
= \sum_j m_j \phi_j^{(\alpha)}(x_1, \ldots, x_N) \sum_n m_n \chi_n^{(\beta)}(\sigma_1, \ldots, \sigma_N) \sum_k m_k D_{jk}^{(\alpha)}(P) D_{nk}^{(\beta)}(P)
\]

one finds that \( P\Psi = (-1)^P \Psi \) needs: \( \sum_k m_k D_{jk}^{(\alpha)}(P) D_{nk}^{(\beta)}(P) = (-)^P \delta_{nj} \).

This is true if \( D_{jk}^{(\alpha)}(P) = (-)^P D_{kj}^{(\beta)}(P^{-1}) \) because then

\[
\sum_k m_k D_{jk}^{(\alpha)}(P) D_{nk}^{(\beta)}(P) = \sum_k (-)^P D_{kj}^{(\beta)}(P^{-1}) D_{nk}^{(\beta)}(P) = (-)^P D_{nj}^{(\beta)}(E)
\]
Recall the Theorem: \( D^{(\alpha)}_{jk}(P) = (-)^P D^{(\beta)}_{kj}(P^{-1}) \) if conjugate irreps.

Spin eigenfunctions can have Young tableaux of 1 or 2 lines since the spin states are only 2. Here is a possible \([N-M,M]\) tableau:

\[
\begin{array}{cccccc}
  a_1 & a_2 & \cdots & \cdots & \cdots & a_{N-M} \\
  b_1 & b_2 & \cdots & b_M \\
\end{array}
\]

Then this is the \([2^M,1^{N-M}]\) conjugate tableau:
**Irreducible Tensor Operators**

Symmetry operators act on wave functions and on any other operator. Let \( \Phi, \Psi \) wavefunctions, suppose \( \Phi = \hat{A} \Psi \), \( \hat{A} \) = some operator. Now let \( R \) be a symmetry, \( R^\dagger R = 1 \).

Acting with \( R \): \( R\Phi = R\hat{A}\Psi \iff R\Phi = R\hat{A} R^\dagger R\Psi \).

So the rules are: \( \Phi \rightarrow R\Phi \), \( \Psi \rightarrow R\Psi \), \( \hat{A} \rightarrow R\hat{A} R^\dagger \).

We can consider \((x_1, x_2, x_3)\) as a set of functions or as the components of an operator, actually the two rules differ by a matter of notation.

\[
x_i \rightarrow Rx_i = D(R)_{ik} x_k \quad \text{if we think of functions}
\]

\[
x_i \rightarrow Rx_i R^\dagger \quad \text{if we think of operators, then}
\]

\( R^\dagger \) just halts the action of \( R \) on the right.

**But the linear combinations are the same!**
( \( x_1, x_2, x_3 \) ) is a vector operator
\[
x_i \rightarrow Rx_i = D(R)_{ik} x_k
\]

More generally, a tensor is a set of components that are sent into linear combinations by every symmetry \( S \) in Group \( G \)

\[
T_i \rightarrow ST_i S^\dagger = \sum_j T_j D_{ji}(S)
\]

The Group multiplication table is OK since

\[
RS : T_i \rightarrow RST_i S^\dagger R^\dagger = R \sum_j T_j D_{ji}(S) R^\dagger = \sum_k T_k D_{kj}(R) D_{ji}(S)
\]

\[
= \sum_k T_k D_{ki}(RS)
\]

\( D \) matrices make a representation of \( G \). If the representation is the irrep \( \alpha \), we can speak of the irreducible tensor operator \( T^{(\alpha)} \)

irreducible implies that all its components are mixed by the Group operations, and one cannot find linear combinations than are not mixed.
In $\text{GL}(n)$ one considers linear transformations in $n$-dimensional space $\mathbb{R}^n$.

A vector $x$ transforms according to a law of the form:

$$x'_i = \sum_{j}^{n} a_{ij} x_j \quad i \in (1,...n) \quad x,x' \in \mathbb{R}^n$$

In the case of multi-index tensors the linear transformation law is the same as if it were a product of coordinates

$$T'_{\alpha(1),\alpha(2)\ldots\alpha(r)} = \sum_{\beta(1),\beta(2)\ldots\beta(r)} a_{\alpha(1)\beta(1)} a_{\alpha(2)\beta(2)}\ldots a_{\alpha(r)\beta(r)} T_{\beta(1),\beta(2)\ldots\beta(r)}$$

$$\alpha(k) \in (1,...n), \beta(k) \in (1,...n).$$

In $\mathbb{R}^3$, from $T_{ij} = x_i x_j$ one obtains the scalar $\text{Tr}T = x_1^2 + x_2^2 + x_3^2$ and other linear combinations that transform like quadrupole tensor components.
Tensors of rank 2

One can build symmetric and antisymmetric tensors of rank 2 (2 indices)

\[ T_{ij} \rightarrow \begin{cases} S_{ij} = T_{ij} + T_{ji} \\ A_{ij} = T_{ij} - T_{ji} \end{cases} \]

Symmetry and antisymmetry are conserved under GL(n) transformations!

\[ T_{ij} \rightarrow a_{im} a_{jn} T_{mn} \]

\[ T_{ji} \rightarrow a_{jm} a_{in} T_{mn} \]

\[ \begin{cases} S_{ij} = T_{ij} + T_{ji} & \rightarrow (a_{jm} a_{in} + a_{jn} a_{im}) T_{mn} \quad \text{symmetric} \\ A_{ij} = T_{ij} - T_{ji} & \rightarrow (a_{jm} a_{in} - a_{jn} a_{im}) T_{mn} \quad \text{antisymmetric} \end{cases} \]

In general, the tensors of GL(n) are reduced into irreducible parts by taking linear combinations according to the irreps of the permutation Group S(r) where r=rank of T. Further reduction is possible when considering subgroups of GL(n).