Conserving Approximations

We can make no use of approximations that violate the basic conservation laws. The continuity equation must hold; energy and momentum conservation laws, when applicable, must also be obeyed. An approximation that respects these fundamental symmetries is called conserving.

The Hartree-Fock approximation is conserving, but if we pick a general $\Sigma$ this is not granted automatically.

\[
\frac{\partial \rho}{\partial t} + \text{div } \vec{j} = 0
\]

Let us see how the Green’s function formalism embodies current conservation.
Part 16 page 22

\[ \text{ig}^{(T)} \sigma, \sigma' (\vec{x}, t, \vec{x}', t') = \left< T \psi_\sigma (\vec{x}, t) \psi^\dagger_{\sigma'} (\vec{x}', t') \right> \]

\[ = \theta (t - t') \left< \psi_\sigma (\vec{x}, t) \psi^\dagger_{\sigma'} (\vec{x}', t') \right> - \theta (t' - t) \left< \psi^\dagger_{\sigma'} (\vec{x}', t') \psi_\sigma (\vec{x}, t) \right> \]

Part 16 page 24

\[ \rho_\sigma (x, t) = -i \lim_{t' \to t+} \lim_{x' \to x} g^{(T)}_{\sigma\sigma} (x, t, x', t') \]

\[ \vec{j}_\sigma (x, t) = \frac{-1}{2m} \lim_{t' \to t+} \lim_{x' \to x} \left( \nabla_x - \nabla_{x'} \right) g^{(T)}_{\sigma\sigma} (x, t, x', t') \]

The Dyson equation has two forms:
\( G(x, x') = G_0(x, x') + \int dx_1 dx_2 G_0(x, x_1) \Sigma(x_1, x_2) G(x_2, x') \)

or \( G = G_0 + G_0 \Sigma G \) in matrix form

and

\[ G(x, x') = G_0(x, x') + \int dx_1 dx_2 G(x, x_1) \Sigma(x_1, x_2) G_0(x_2, x') \]

\[ G = G_0 + G \Sigma G_0. \]

We saw that:

\[ G(x, x') = G_0(x, x') + \int dx_1 dx_2 G_0(x, x_1) \Sigma(x_1, x_2) G(x_2, x') \]

\[ \Rightarrow \text{Differential form of Dyson equation:} \]

\[ \left[ i \frac{\partial}{\partial t} - h_0(x) \right] G(x, x') = \delta(x - x') \delta(t - t') + \int dx_2 \Sigma(x, x_2) G(x_2, x'), \]
while
\[ G(x, x') = G_0(x, x') + \int dx_1 dx_2 G(x, x_1) \Sigma(x_1, x_2) G_0(x_2, x') \]
⇒ another differential form of Dyson equation:
\[
\left[ -i \frac{\partial}{\partial t'} - h_0(x') \right] G(x, x') = \delta(x - x') \delta(t - t') + \int dx_2 G(x, x_2) \Sigma(x_2, x').
\]

Let \( h_0(x) = -\frac{\nabla^2}{2m} + U(x) \) denote the free part of \( H \)
and simplify notation with \( x \rightarrow 1, x' \rightarrow 2. \)

The two differential forms become
\[
\begin{align*}
[i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1)] \hat{G}(1, 2) &= \delta(1 - 2) + \int d\Sigma(1, \bar{1}) G(\bar{1}, 2). \\
[-i \frac{\partial}{\partial t_2} + \frac{\nabla_2^2}{2m} - U(2)] \hat{G}(1, 2) &= \delta(1 - 2) + \int d\bar{1} G(1, \bar{1}) \Sigma(\bar{1}, 2)
\end{align*}
\]
From

\[ i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U(1) \hat{G}(1, 2) = \delta(1 - 2) + \int d\Sigma(1, \bar{1}) G(\bar{1}, 2). \]

\[ -i \frac{\partial}{\partial t_2} + \frac{\nabla_2^2}{2m} - U(2) \hat{G}(1, 2) = \delta(1 - 2) + \int d\bar{1} G(1, \bar{1}) \Sigma(\bar{1}, 2) \]

taking the difference, since \( \nabla_1^2 - \nabla_2^2 = (\nabla_1 + \nabla_2)(\nabla_1 - \nabla_2) \)

\[(i \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2})G(1, 2) - (\nabla_1 + \nabla_2)(\nabla_1 - \nabla_2) \frac{G(1, 2)}{2m} = [U(1) - U(2)] G(1, 2) + Z(1, 2) \]

where

\[ Z(1, 2) = \int d\bar{1} \left\{ \Sigma(1, \bar{1}) G(\bar{1}, 2) - G(1, \bar{1}) \Sigma(\bar{1}, 2) \right\}. \]

Now I show that the continuity equation requires: \( Z(1, 1^+) = 0. \)

that is,

\[ Z(1, 1^+) = \int d\bar{1} \left\{ \Sigma(1, \bar{1}) G(\bar{1}, 1^+) - G(1, \bar{1}) \Sigma(\bar{1}, 1^+) \right\} = 0. \]
To show this, start from
\[ (i \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2}) G(1, 2) - (\nabla_1 + \nabla_2) (\nabla_1 - \nabla_2) \frac{G(1, 2)}{2m} = [U(1) - U(2)] G(1, 2) + Z(1, 2) \]
and consider the appropriate limiting case \( 2 \to 1^+ \).

By definition, \( iG(1, 2) = \langle \Psi_0 | T[\psi_{1H} (t_1) \psi_{2H}^\dagger (t_2)] | \Psi_0 \rangle \)
\[ = \langle \Psi_0 | \theta(t_1 - t_2) \psi_{1H} (t_1) \psi_{2H}^\dagger (0) - \theta(t_2 - t_1) \psi_{2H}^\dagger (t_2) \psi_{1H} (t_1) | \Psi_0 \rangle, \]
with \( | \Psi_0 \rangle = \text{interacting g.s.} \)
\[ iG(1, 1^+) = - \langle \Psi_0 | \psi_{2H}^\dagger (t + 0) \psi_{1H} (t) | \Psi_0 \rangle_{2=1^+} \]
set \( 2 = 1^+ \) i.e. \( t_2 \) just after \( t_1 \), with the same \( x \). Since \( iG(1, 1^+) = - \langle \Psi^\dagger (1) \Psi (1) \rangle = - \langle n(1) \rangle, \) where \( n = \text{density}, \)
\[ \left[ \frac{\nabla_1 - \nabla_2}{2m} G(1, 2) \right]_{2=1^+} = - \frac{\langle \Psi^\dagger (1) \nabla_1 \Psi (1) - (\nabla_1 \Psi^\dagger (1)) \Psi (1) \rangle}{2mi} = - \langle j(1) \rangle \]
Putting together
\[
(i \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2})G(1, 2) + (\nabla_1 + \nabla_2)(\nabla_1 - \nabla_2) \frac{G(1, 2)}{2m} = [U(1) - U(2)]G(1, 2) + Z(1, 2)
\]

and \(iG(1, 1^+) = -\langle \Psi^+(1)\Psi(1) \rangle = -\langle n(1) \rangle\), where \(n = \) density,
\[
\left[ \frac{\nabla_1 - \nabla_2}{2m} G(1, 2) \right]_{2=1^+} = -\langle j(1) \rangle
\]

one finds:
\[
-2 \frac{\partial}{\partial t_1} \langle n(1) \rangle - 2\nabla_1 \langle j(1) \rangle = [U(1) - U(1^+)]G(1, 1^+) + Z(1, 1^+) = Z(1, 1^+)
\]

The Dyson equation takes here, but continuity equation \(\Rightarrow Z(1, 1^+) = 0\),
which is not granted for arbitrary approximations!
\[
Z(1, 1^+) = \int d\bar{1} \{ \Sigma(1, \bar{1})G(\bar{1}, 1^+) - G(1, \bar{1})\Sigma(\bar{1}, 1^+) \} = 0 \text{ for the exact } \Sigma
\]

The exact self-energy must imply that!
Let me summarize.

Continuity: 
\[-2 \frac{\partial}{\partial t_1} \langle n(1) \rangle - 2 \nabla_1 \langle j(1) \rangle = Z(1, 1^+) = 0\]

where

\[Z(1, 2) = \int d\bar{1} \left\{ \Sigma(1, \bar{1})G(\bar{1}, 2) - G(1, \bar{1})\Sigma(\bar{1}, 2) \right\}\]

Condition on approximations, which can only be tenable if they are conserving.

*The condition on approximate \( \Sigma \) and \( g \) is that not only both forms* 

\[G = G_0 + G_0 \Sigma G\quad G = G_0 + G \Sigma G_0\]

of the Dyson equation be obeyed but also \( Z(1, 1^+) = 0 \).
Since

\[ \begin{array}{c}
\begin{array}{c}
\text{=}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{=}
\end{array}
\end{array} \]

It is clear that

\[ \begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\text{=}
\end{array}
\begin{array}{c}
2
\end{array}
\end{array} \]

But the condition \( Z(1,1^+) = 0 \) implies that

This leads us to consider a new functional

Next, we discuss how to build conserving approximations.
The $\Phi$ Functional

There is a simple diagrammatic prescription for creating conserving approximations. Let $\delta \Sigma(n)_{skel}$ denote a skeleton diagram of order $n$ for the self-energy $\Sigma(1, 2)$ and imagine joining the two ends by an exact propagator $g(2, 1)$. The result is the ring diagram in the next Figure:

$$\delta \Phi^{(n)} = \frac{\int d1d2 \Sigma(1, 2) g(2, 1)}{2n}.$$  

This is similar to one of those bubble diagrams that contribute to the no-particle propagator or ground-state energy, except that the $2n$ internal propagators are dressed $g$ and there are no self-energy insertions. The analytical expression of any diagram is a functional of $g$. Any graph with $2n$ $g$ lines must be divided by $2n$. The reason is that we wish to recover the skeleton diagram by a functional differentiation of F and the same skeleton appears $2n$ times.
the inverse operation (opening the \( g(2, 1) \) line to restore the original self-energy) is a functional differentiation.

The functional derivative

\[
\begin{align*}
\sum (1, 2) &= \frac{\delta \Sigma (1,2)}{\delta g(4,5)}
\end{align*}
\]

Fig. 11.25. Diagrammatic interpretation of the functional derivative.
Examples of the recovering of self-energy skeleton diagrams

We also introduce an interaction strength parameter $\lambda$ (which is 1 in the fully interacting case).
Figure 6: $\Phi$ and the corresponding self-energies $\Sigma$ for (a) the self-consistent Hartree-Fock approximation, (b) the self-consistent T-matrix approximation, (c) the shielded potential approximation.

Figure from a talk by Gordon Baym
Let $\sigma_n$ represent the sum of all the skeleton diagrams with $n$ vertices; the $\Phi$ functional is:

$$\Phi[g, \lambda] = \sum_n \frac{\lambda^n}{2n} Tr[g(\omega)\sigma_n(\omega)],$$

where

$$Tr = \sum_{\text{spin}} \sum_{\text{internal labels}} \int \frac{d\omega}{2\pi}$$

as implied by the diagram rules.
Luttinger-Ward theorem
The exact self-energy is given by (note the order of arguments!)

\[ \Sigma(1, 2) = \frac{\delta \Phi}{\delta g(2,1)} \]

The continuity equation and the momentum, angular momentum and energy conservation laws are embodied in the \( \Phi \) functional.

**Baym and Kadanoff Theorem**

If (and only if) a self-energy is \( \Phi \)-derivable, that is, comes by functional differentiation from some approximate \( \Phi \), the approximation is conserving.

In other terms, even for approximate \( \Sigma \), the property of \( \Phi \) derivability is equivalent to the conservation laws.

The HF and GW approximations are \( \Phi \)-derivable and hence conserving.
Proof of the Baym and Kadanoff Theorem

If (and only if) a self-energy is $\Phi$-derivable, that is, comes by functional differentiation from some approximate $\Phi$, the approximation implies the continuity equation (similar proofs for momentum, angular momentum and energy).

Under a gauge transformation the field operator $\Psi(x) \rightarrow \Psi(x)e^{i\Lambda(x)}$,

the time-ordered Green's function $iG^T(1, 2) = \left\langle T\psi(1)\psi^\dagger(2) \right\rangle$

undergoes the transformation $G(1, 2) \rightarrow e^{i\Lambda(1)}G(1, 2)e^{-i\Lambda(2)}$.  

---

Leo Philip Kadanoff è stato un fisico statunitense. È conosciuto per i suoi lavori in fisica statistica, in fisica della materia condensata e in teoria del caos.

Wikipedia
In an infinitesimal gauge transformation $\delta \Lambda$, the change is:

$$
\delta G(1, 2) = e^{i(\delta \Lambda(1) - \delta \Lambda(2))} G(1, 2) - G(1, 2) \approx i \left[ \delta \Lambda(1) - \delta \Lambda(2) \right] G(1, 2)
$$

$\Phi$ is invariant, since for any line entering a vertex there is another one leaving it. So we have a relation $\delta \Phi = 0$. On the other hand all lines enter symmetrically and must give the same contribution, i.e. 0. Therefore, the change of the functional under a gauge transformation is:
\[ 0 = \delta \Phi = \int d1d2 \frac{\delta \Phi}{\delta G(1,2)} \delta G(1,2) = \int d1d2 \Sigma(1,2) \delta G(1,2). \]

Insert \( \delta G(1,2) = i(\delta \Lambda(1) - \delta \Lambda(2))G(1,2) \)

\[ 0 = \int d1d2 \Sigma(1,2)(\delta \Lambda(1) - \delta \Lambda(2))G(1,2). \]

Exchanging dummy variables in the second term,

\[ 0 = \int d1d2 \Sigma(1,2)(\delta \Lambda(1)G(1,2) - \delta \Lambda(1)G(2,1)) = \]

\[ = \int d1d2[\Sigma(2,1)G(1,2) - G(2,1)\Sigma(1,2)]\delta \Lambda(1) \quad \forall \delta \Lambda(1) \]

\[ \Rightarrow \int d2 (G(1,2)\Sigma(1,2) - \Sigma(1,2)G(2,1)) = 0 \]

\[ Z(1,1) = 0 \]

where \( Z(1,1^+) = \int d\bar{T}\left\{\Sigma(1, \bar{T})G(\bar{T}, 1^+) - G(1, \bar{T})\Sigma(\bar{T}, 1^+)\right\} \).

As shown above, this condition ensures that the approximation is conserving.
Ward identities

An exact relation between $g$ and $G_2$ results from the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div} \; \vec{j} = 0, \quad \text{where}$$

$$\rho = \psi^\dagger(x)\psi(x) \quad j_i(x) = \frac{\hbar}{2mi} \left[ \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i'} \right) \psi^\dagger(x')\psi(x) \right]_{x' = x}$$

It is:

$$\frac{\partial}{\partial t} G_2(x, x_1, x_2, x') + \frac{\hbar}{2mi} \nabla \left( \nabla - \nabla' \right) G_2(x, x_1, x_2, x')$$

$$= -i\delta^{(4)} \left( x - x_1 \right) g(x, x_2) + i\delta^{(4)} \left( x - x_2 \right) g(x_1, x)$$

The continuity equation is an expression of charge conservation and is associated by Noether’s theorem to the gauge invariance. Ward identities generally arise from invariance Groups of the theory.
Strategy of the Proof

Work out the identity

\[
\frac{d}{dt} \langle T \{ \psi(x_1)\psi^\dagger(x_2)\rho(t) \} \rangle = \langle T \{ \psi(x_1)\psi^\dagger(x_2) \frac{d}{dt} \rho(t) \} \rangle
\]

\[-\delta^{(4)}(x - x_1)ig(x, x_2) + \delta^{(4)}(x - x_2)ig(x_1, x).\]

Insert \( \frac{\partial \rho}{\partial t} + \text{div } \vec{j} = 0 \) \( \rho = \psi^\dagger(x)\psi(x) \)

Simplify
Proof

Let $A_1, A_2$ denote Heisenberg operators such as Fermion creation or annihilation operators and let $\rho$ be an operator such as a density.

First, let us show that, from definition of Wick’s T operator,\

$$\frac{d}{dt} T \{ A_1(t_1) A_2(t_2) \rho(t) \} = T \left\{ A_1(t_1) A_2(t_2) \frac{\partial}{\partial t} \rho(t) \right\}$$

$$+ \delta(t - t_1) T \{ [\rho(t), A_1(t_1) - A_2(t_2)] \} + \delta(t - t_2) T \{ A_1(t_1) [\rho(t), A_2(t_2)] \}$$

Under Wick’s T ordering $\rho$ commutes with $A_1, A_2$.

We consider separately the two possible orderings of $t_1 > t_2$ and $t_1 < t_2$. 

21
For $t_1 > t_2$, $A_1(t_1)$ on the left of $A_2(t_2)$, one has 3 orderings:

For $t_1 > t_2$, $A_1(t_1)$ on the left of $A_2(t_2)$, one has 3 orderings:

\[
T \left\{ A_1(t_1) A_2(t_2) \rho(t) \right\} =
\]
\[
\theta(t - t_1) \rho(t) A_1(t_1) A_2(t_2)
\]
\[
+ \theta(t_1 - t) \theta(t - t_2) A_1(t_1) \rho(t) A_2(t_2)
\]
\[
+ \theta(t_2 - t) A_1(t_1) A_2(t_2) \rho(t)
\]

insert theta functions
\[ T \{ A_1(t_1) A_2(t_2) \rho(t) \} = \theta(t - t_1) \rho(t) A_1(t_1) A_2(t_2) \]
\[ + \theta(t_1 - t) \theta(t - t_2) A_1(t_1) \rho(t) A_2(t_2) + \theta(t_2 - t) A_1(t_1) A_2(t_2) \rho(t). \]

\[ \frac{d}{dt} \] yields one contribution from \( \rho(t) \) and 4 from the \( \theta \) functions.

\[ \frac{d}{dt} \rightarrow T \left\{ A_1(t_1) A_2(t_2) \frac{\partial \rho(t)}{\partial t} \right\} + \delta(t - t_1) \rho(t) A_1(t_1) A_2(t_2) \]
\[ - \delta(t - t_1) \theta(t - t_2) A_1(t_1) \rho(t) A_2(t_2) \]
\[ + \delta(t - t_2) \theta(t - t_2) A_1(t_1) \rho(t) A_2(t_2) - \delta(t_2 - t) A_1(t_1) A_2(t_2) \rho(t) \]
\[ = T \left\{ A_1(t_1) A_2(t_2) \frac{\partial \rho(t)}{\partial t} + \delta(t - t_1) [\rho(t), A_1(t_1)] A_2(t_2) + \delta(t - t_2) A_1(t_1) [\rho(t), A_2(t_2)] \right\}. \]

We assumed \( t_1 > t_2 \), \textit{but} for \( t_1 < t_2 \), the result is the same.
Since the result is symmetric in $t_1$ and $t_2$, and for $t_2 > t_1$ is the same:

\[
\frac{d}{dt} T \{ A_1(t_1) A_2(t_2) \rho(t) \} = T \left\{ A_1(t_1) A_2(t_2) \frac{\partial}{\partial t} \rho(t) \right\} \\
+ \delta(t-t_1) T \{ [\rho(t), A_1(t_1)]_\text{\textminus} A_2(t_2) \} + \delta(t-t_2) T \{ A_1(t_1)[\rho(t), A_2(t_2)]_\text{\textminus} \}
\]

QED,

Now it is apparent that if we set \( A_1(t_1) = \psi(x_1), \quad A_2(t_2) = \psi^\dagger(x_2) \) and \( \rho(x) = \psi^\dagger(x) \psi(x) \) this is a 2-body operator, but the commutators will give deltas times one-body operators. So essentially we are almost finished.
In the identity \( \frac{d}{dt} T \{ A_1(t_1) A_2(t_2) \rho(t) \} = T \left\{ A_1(t_1) A_2(t_2) \frac{d}{dt} \rho(t) \right\} \)
\[ + \delta(t - t_1) T \{ [\rho(t), A_1(t_1)] - A_2(t_2) \} + \delta(t - t_2) T \{ A_1(t_1) [\rho(t), A_2(t_2)] \} \]

set \( A_1(t_1) = \psi(x_1), \quad A_2(t_2) = \psi^\dagger(x_2) : \)

\[ \frac{d}{dt} T \{ \psi(x_1)\psi^\dagger(x_2) \rho(t) \} = T \left\{ \psi(x_1)\psi^\dagger(x_2) \frac{\partial}{\partial t} \rho(t) \right\} \]
\[ + \delta(t - t_1) T \{ [\rho(t), \psi(x_1)] - \psi^\dagger(x_2) \} + \delta(t - t_2) T \{ \psi(x_1) [\rho(t), \psi^\dagger(x_2)] \}. \]

Now one must work out the commutators 
\[ [\rho(t), \psi(x_1)] \quad \text{and} \quad [\rho(t), \psi^\dagger(x_2)]. \]
Here, since $\rho = \psi^\dagger(x)\psi(x)$,

$$[\rho(x), \psi(x_1)]_\_ = \psi^\dagger(x)\psi(x)\psi(x_1) - \psi(x_1)\psi^\dagger(x)\psi(x)$$

$$= -\psi^\dagger(x)\psi(x_1)\psi(x) - \psi(x_1)\psi^\dagger(x)\psi(x) =$$

$$- (\psi^\dagger(x)\psi(x_1) + \psi(x_1)\psi^\dagger(x)) \psi(x)$$

and anticommutation rules give

$$[\rho(x), \psi(x_1)]_\_ = -\delta(x - x_1)\psi(x_1);$$

in the same way,

$$[\rho, \psi^\dagger(x_2)]_\_ = \delta(x - x_2)\psi^\dagger(x_2)$$
Substitute \( [\rho(x), \psi(x_1)]_\gamma = -\delta(x - x_1)\psi(x) \), and \( [\rho, \psi^\dagger(x_2)]_\gamma = \delta(x - x_2)\psi^\dagger(x) \)

into

\[
\begin{align*}
\frac{d}{dt} T \left\{ \psi(x_1)\psi^\dagger(x_2) \rho(t) \right\} &= T \left\{ \psi(x_1)\psi^\dagger(x_2) \frac{\partial}{\partial t} \rho(t) \right\} \\
+ \delta(t - t_1) T \left\{ [\rho(t), \psi(x_1)]_\gamma \psi^\dagger(x_2) \right\} + \delta(t - t_2) T \left\{ \psi(x_1)[\rho(t), \psi^\dagger(x_2)]_\gamma \right\}
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} T \left\{ \psi(x_1)\psi^\dagger(x_2) \rho(t) \right\} &= T \left\{ \psi(x_1)\psi^\dagger(x_2) \frac{\partial}{\partial t} \rho(t) \right\} \\
+ \delta(t - t_1) T \left\{ (-\delta(x - x_1)\psi(x)) \psi^\dagger(x_2) \right\} + \delta(t - t_2) T \left\{ \psi(x_1)\delta(x - x_2)\psi^\dagger(x) \right\}
\end{align*}
\]
Next, re-order and average

\[
\frac{d}{dt} T \left\{ \psi(x_1)\psi^\dagger(x_2) \rho(t) \right\} = T \left\{ \psi(x_1)\psi^\dagger(x_2) \frac{d}{dt} \rho(t) \right\} \\
+ \delta(t-t_1)T \left\{ (-\delta(\bar{x} - \bar{x}_1)\psi(x))\psi^\dagger(x_2) \right\} + \delta(t-t_2)T \left\{ \psi(x_1)\delta(\bar{x} - \bar{x}_2)\psi^\dagger(x) \right\}
\]

\[
\frac{d}{dt} \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2) \rho(t) \right\} \right\rangle = \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2) \frac{d}{dt} \rho(t) \right\} \right\rangle \\
- \delta^{(4)}(x - x_1) \left\langle T \left\{ (\psi(x))\psi^\dagger(x_2) \right\} \right\rangle + \delta^{(4)}(x - x_2) \left\langle T \left\{ \psi(x_1)\psi^\dagger(x) \right\} \right\rangle
\]

and since \( g(x_1x_2) = -i \left\langle T\psi(x_1)\psi^\dagger(x_2) \right\rangle \)

\[
\frac{d}{dt} \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2) \rho(t) \right\} \right\rangle = \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2) \frac{d}{dt} \rho(t) \right\} \right\rangle \\
- \delta^{(4)}(x - x_1)ig(x, x_2) + \delta^{(4)}(x - x_2)ig(x_1, x)
\]
We got the identity

\[
\frac{d}{dt} \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2)\rho(t) \right\} \rightangle = \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2) \frac{d}{dt} \rho(t) \right\} \rightangle 
- \delta^{(4)}(x - x_1)ig(x, x_2) + \delta^{(4)}(x - x_2)ig(x_1, x).
\]

Here physics enters: insert \( \frac{\partial \rho}{\partial t} + \text{div} \, \vec{j} = 0 \quad \rho = \psi^\dagger(x)\psi(x) \)

\[
\frac{d}{dt} \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2)\psi^\dagger(x)\psi(x) \right\} \rightangle = -\nabla \left\langle T \left\{ \psi(x_1)\psi^\dagger(x_2) \, j(x) \right\} \rightangle 
- \delta^{(4)}(x - x_1)ig(x, x_2) + \delta^{(4)}(x - x_2)ig(x_1, x);
\]
the l.h.s. is a two-body stuff!
Introduce the two-particle GF

\[ G_2(x_1, x_2, x_3, x_4) = -\langle T \left[ \psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4) \right] \rangle \]

\[
\frac{d}{dt}\langle T \left\{ \psi(x_1)\psi(x)\psi^\dagger(x_2)\psi^\dagger(x) \right\} \rangle = -\nabla \langle T \left\{ \psi(x_1)\psi^\dagger(x_2) j(x) \right\} \rangle
\]

\[-\delta^{(4)}(x-x_1)ig(x, x_2) + \delta^{(4)}(x-x_2)ig(x_1, x)\]

Substitute \( j_i(x) = \frac{\hbar}{2mi} \left[ \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x'_i} \right) \psi^\dagger(x')\psi(x) \right]_{x'=x} \)

\[
\frac{d}{dt} G_2(x, x_1, x_2, x) = -\frac{\hbar}{2mi} \frac{\partial}{\partial x_i} \left[ \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x'_i} \right) \langle T \left\{ \psi(x_1)\psi^\dagger(x_2)\psi^\dagger(x')\psi(x) \right\} \rangle \right]_{x'=x}
\]

\[-\delta^{(4)}(x-x_1)ig(x, x_2) + \delta^{(4)}(x-x_2)ig(x_1, x)\]
\[
\frac{d}{dt} G_2(x, x_1, x_2, x) + \frac{\hbar}{2mi} \frac{\partial}{\partial x_i} \left[ \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x'_i} \right) \langle T \{ \psi(x_1) \psi(x) \psi^\dagger(x_2) \psi^\dagger(x') \} \rangle \right]_{x'=x} \\
= -\delta^{(4)}(x-x_1)ig(x, x_2) + \delta^{(4)}(x-x_2)ig(x_1, x)
\]

Ward identity: for \( x' \to x \),

\[
\frac{\partial}{\partial t} G_2(x, x_1, x_2, x') + \frac{\hbar}{2mi} \nabla \left( \nabla - \nabla' \right) G_2(x, x_1, x_2, x') \\
= -i\delta^{(4)}(x-x_1)g(x, x_2) + i\delta^{(4)}(x-x_2)g(x_1, x)
\]
Consider the vertex

\[ \Gamma(1, 2, 3, k, q, \alpha, \omega) = \]

In homogeneous systems one can show that

\[
\lim(q \to 0, \alpha \to 0, \frac{q}{\alpha} \to 0) \quad \Gamma(1, 2, 3, k, q, \alpha, \omega) = 1 + \frac{\partial \Sigma(k, \omega)}{\partial \omega} \]

\[
\lim(q \to 0, \alpha \to 0, \frac{q}{\alpha} \to \infty) \quad \Gamma(1, 2, 3, k, q, \alpha, \omega) = 1 + \frac{\partial \Sigma(k, \omega)}{\partial \omega} + \frac{\partial \Sigma(k, \omega)}{\partial \mu} \]

Other Ward identities are known (see William Jones, Norman March ‘Theoretical Solid State Physics’ Dover)
Herglotz property: the DOS must be positive!