Relating Gauge Theories via the Gauge/Bethe Correspondence

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based on arXiv:1005.4445, 1011.6120 and work in progress
with D. Orlando
Intro/Summary

Use techniques from integrable models to relate susy gauge theories.

Tool: Gauge/Bethe correspondence as stated by Nekrasov/Shatashvili.

Two parts: specific example underlying framework

Example: 3 different $N=(2,2)$ quiver gauge theories in 2d have the same susy ground states:

Correspondence to an integrable spin chain: $tJ$ model

Can apply this technique in a general context.
Intro/Summary

Underlying framework: until now, the reason why the Gauge/Bethe correspondence works has not been understood.

The integrable system side of the correspondence teaches us that it makes sense to study U(N) gauge theories with different N in a unified fashion.

Moreover, these gauge theories carry a symmetry group such as e.g. SU(2)

A mathematical framework that connects different gauge theories as described above exists:

geometric representation theory

Underlying reason for the Gauge/Bethe correspondence!
Outline

* Gauge/Bethe correspondence
* Example:
  * The tJ model
  * Supergroup symmetry
  * Bethe ansatz
  * The Dictionary
  * Quiver gauge theories
  * Relation via brane cartoons
* The big picture
* Summary
The Gauge/Bethe Correspondence

Relates $N=(2,2)$ gauge theories in 2d to integrable spin chains.
The susy vacua of the gauge theory correspond to the Bethe spectrum of the spin chain.
Generators of chiral ring correspond to commuting Hamiltonians.
Integrable model: spectrum determined by Bethe equations.
Gauge theory: ground states determined by eff. twisted superpotential.
Correspondence works for all Bethe solvable spin chains.
Spin chains with supergroup symmetry correspond to quiver gauge theories.

Nekrasov, Shatashvili
The Gauge/Bethe Correspondence

$N=(2,2)$ gauge theories in 2d:

**Vector multiplet:**

\[ V = \theta^{-}\bar{\theta}^{-}(A_0 - A_1) + \theta^{+}\bar{\theta}^{+}(A_0 + A_1) - \theta^{-}\bar{\theta}^{+}\sigma - \theta^{+}\bar{\theta}^{-}\bar{\sigma} \]
\[ + i\theta^{-}\theta^{+}(\bar{\theta}^{-}\bar{\lambda}_- + \bar{\theta}^{+}\bar{\lambda}_+) + i\bar{\theta}^{+}\bar{\theta}^{-}(\theta^{-}\lambda_- + \theta^{+}\lambda_+) + \theta^{-}\theta^{+}\bar{\theta}^{-}D \]

**Chiral multiplet:**

\[ \Phi = \phi(y^\pm) + \theta^\alpha \psi^\alpha(y^\pm) + \theta^{+}\theta^{-}F(y^\pm) \]
\[ y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm \]
\[ x^\pm = x_0 \pm x^1 \]

**Twisted chiral multiplet:**

\[ \Sigma = \sigma(\tilde{y}^\pm) + i\theta^+\bar{\lambda}_+(\tilde{y}^\pm) - i\bar{\theta}^-\lambda_-(\tilde{y}^\pm) + \theta^+\bar{\theta}^-[D(\tilde{y}^\pm) - iA_{01}(\tilde{y}^\pm)] + ... \]
\[ A_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] \]
\[ \tilde{y}^\pm = x^\pm \mp i\theta^\pm\bar{\theta}^\pm \]
The Gauge/Bethe Correspondence

Action: D terms, F terms, twisted F terms

Twisted F term: \[ \int d^2 x \, d\bar{\theta}^- \, d\theta^+ \, \tilde{W} \bigg|_{\bar{\theta}^+ = \theta^- = 0} + \text{h.c.} \]

Kinetic term of action:
\[ L_{\text{kin}} = \int d^4 \theta \left( \sum_k X_k^\dagger \, e^V \, X_k - \frac{1}{2e^2} \text{Tr}(\Sigma^\dagger \Sigma) \right), \]

Twisted masses:
\[ L_{\text{tw}} = \int d^4 \theta \left( X^\dagger \, e^{\theta^-} \bar{\theta}^+ \bar{m} \, X + \text{h.c.} \, X \right) \]

Want to consider the Coulomb branch.

Calculate eff. action for slowly varying \( \sigma \) fields

Integrate out all massive matter fields.
The Gauge/Bethe Correspondence

Most general action (at most 4 fermions, 2 derivatives):

\[ S_{\text{eff}}(\Sigma) = -\int d^4\theta \, K_{\text{eff}}(\Sigma, \overline{\Sigma}) + \frac{1}{2} \int d^2\theta \, \tilde{W}_{\text{eff}}(\Sigma) + \text{h.c.} . \]

Integrate out massive fields (S is quadratic in Q)

\[ e^{iS_{\text{eff}}(\Sigma)} = \int DQ \, e^{iS(\Sigma, Q)} \]

This calculation is exact (protected by supersymmetry).

\[ \tilde{W}_{\text{eff}}(\Sigma) = \frac{1}{2\pi} (\Sigma - \tilde{m}_Q) (\log(\Sigma - \tilde{m}_Q) - 1) - \imath \tau \Sigma \]

Vacuum equation:

\[ \exp \left[ 2\pi \frac{\partial \tilde{W}_{\text{eff}}(\sigma)}{\partial \sigma_i} \right] = 1 \]
The Gauge/Bethe Correspondence

General case: quiver gauge theories

What are the parameters?

Vacuum equation:

\[
\exp \left[ 2\pi \frac{\partial \tilde{W}_{\text{eff}}(\sigma)}{\partial \sigma_i} \right] = 1
\]
The Gauge/Bethe Correspondence

What are the parameters of a spin chain?

- boundary conditions
- length of chain
- number of particle species
- symmetry group
- inhomogeneities
- representation
- rank of symm. group
- Cartan

Spectrum is given by solutions of

\[ e^{2\pi i d Y(\lambda)} = 1 \]

Yang counting fn (potential for Bethe equations)
### The Gauge/Bethe Correspondence

<table>
<thead>
<tr>
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<td>FI–term for $U(1)$–factor of gauge group $U(N_a)$</td>
<td>$\tau_a$</td>
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</table>

**Diagram:**

- $U(N_a)$
- $U(L_a)$
- $U(N_b)$
- $U(L_b)$

- $\frac{i}{2} C_{aa}$
- $\frac{i}{2} C_{ab}$
- $\frac{i}{2} C_{bb}$
- $\frac{i}{2} A_k^a + v_k^{(a)}$
- $\frac{i}{2} A_k^a - v_k^{(a)}$
- $\frac{i}{2} A_k^b + v_k^{(b)}$
- $\frac{i}{2} A_k^b - v_k^{(b)}$

*Freitag, 17. Dezember 2010*
tJ-model
The tJ model

Properties:

* spin chain of length L
* periodic boundary conditions
* 3 particle species: spin up, spin down, empty
* $sl(1|2)$ symmetry
* most important feature: 3 inequivalent choices of Cartan matrix

Essler, Korepin

* 3 different (but equivalent) sets of Bethe equations
* spectrum the same for all 3 cases
* fundamental rep. (spin 1/2) at each lattice point
* no inhomogeneities
The tJ model

Electrons on a lattice of length $L$, $\uparrow, \downarrow, \circ$

Hilbert space: $\mathcal{H}_k = \mathbb{C}^{(1|2)}$

Fundamental representation of $sl(1|2)$

creation/annihilation operators: $c_{k,s}^\dagger, c_{k,s}$, $s = \{\uparrow, \downarrow\}$

$$|s\rangle_k = c_{k,s}^\dagger |\circ\rangle_k$$

projector on single occupancy

$$S_k^- = c_{k,\uparrow}^\dagger c_{k,\downarrow}, \quad S_k^+ = c_{k,\downarrow}^\dagger c_{k,\uparrow}, \quad S_k^z = \frac{1}{2} (n_{k,\uparrow} - n_{k,\downarrow})$$

$$n_{k,s} = c_{k,s}^\dagger c_{k,s}$$

$$n_k = n_{k,\uparrow} + n_{k,\downarrow}$$

$$\mathcal{H} = \sum_{k=1}^{L-1} \left[ -t \sum_{s=\uparrow,\downarrow} (c_{k,s}^\dagger c_{k+1,s} + \text{h.c.}) + J \left( S_k \cdot S_{k+1} - \frac{1}{4} n_k n_{k+1} + 2 n_k - \frac{1}{2} \right) \right]$$

nearest neighbor hopping

spin interaction

Freitag, 17. Dezember 2010
The tJ model

Number of holes, up and down spins:

\[ N_h = \sum_{k=1}^{L} (1 - n_k), \quad N_\uparrow = \sum_{k=1}^{L} n_{k,\uparrow}, \quad N_\downarrow = \sum_{k=1}^{L} n_{k,\downarrow}, \quad N_e = N_\uparrow + N_\downarrow. \]

Single occupancy:

\[ L = N_h + N_\uparrow + N_\downarrow \]

tJ model has supergroup symmetry!

A superalgebra can be decomposed into an even and an odd part: \( g = g_0 \oplus g_1 \)

The even part of \( sl(1|2) \) is \( g_0 = gl(1) \oplus sl(2) \) and is generated by \( S^\pm, S^z, Z \)

Fermionic generators: \( Q_s^\pm, s = \{\uparrow, \downarrow\} \)

\[ S_k^- = c_{k,\uparrow}^\dagger c_{k,\downarrow}, \quad S_k^+ = c_{k,\downarrow}^\dagger c_{k,\uparrow}, \quad S_k^z = \frac{1}{2} (n_{k,\uparrow} - n_{k,\downarrow}), \]

\[ Q_{k,\downarrow}^+ = (1 - n_{k,\uparrow}) c_{k,\downarrow}^\dagger, \quad Z_k = 1 - \frac{1}{2} n_k. \]

\[ Q_{k,\uparrow}^- = (1 - n_{k,\downarrow}) c_{k,\uparrow}, \quad Q_{k,\uparrow}^+ = (1 - n_{k,\downarrow}) c_{k,\uparrow}^\dagger, \quad Q_{k,\downarrow}^- = (1 - n_{k,\uparrow}) c_{k,\downarrow}. \]
Supergroup Symmetry

Root decomposition. Lie algebra: reflections of positive root systems are conjugate to each other.

Superalgebra: reflections of odd roots lead to new positive root system (not conjugate).

Here: three choices for Cartan matrix:

<table>
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<tr>
<th>$C^{ab}$</th>
<th>Dynkin diagram</th>
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<tbody>
<tr>
<td>(0, -1)</td>
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<td>(-1, 2)</td>
<td></td>
</tr>
<tr>
<td>(0, -1)</td>
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<tr>
<td>(-1, 0)</td>
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</tr>
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<td></td>
</tr>
<tr>
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odd root

even root
Bethe ansatz

Case A:

\[ C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \Lambda = [0 \ 1], \quad N_1 = N_h, \quad N_2 = N_h + N_\downarrow \]

Nested Bethe ansatz equations:

\[
\left( \frac{\lambda_i^{(a)} + \frac{i}{2} \Lambda^a}{\lambda_i^{(a)} - \frac{i}{2} \Lambda^a} \right)^L = \prod_{\substack{(b,j) = (1,1) \\ (b,j) \neq (a,i)}} \frac{\lambda_i^{(a)} - \lambda_j^{(b)} + \frac{i}{2} C^{ab}}{\lambda_i^{(a)} - \lambda_j^{(b)} - \frac{i}{2} C^{ab}}, \quad a = 1, 2, \ldots, r, \quad i = 1, 2, \ldots, N_a
\]

Yang Yang counting function:

\[
Y_A(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \hat{x}(2\lambda_p^{(2)}) - \frac{1}{2\pi} \sum_{p,q=1}^{N_h+N_\downarrow} \hat{x}(\lambda_p^{(2)} - \lambda_q^{(2)}) + \frac{1}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \sum_{i=1}^{N_h} \hat{x}(2\lambda_p^{(2)} - 2\lambda_i^{(1)})
\]

\[
\hat{x}(\lambda) = \lambda \arctan(\lambda^{-1}) + \frac{1}{2} \log(1 + \lambda^2) + \sum_{i=1}^{N_h} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_h+N_\downarrow} n_p^{(2)} \lambda_p^{(2)}
\]
Bethe ansatz

Case A:

\[ C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad N_1 = N_h, \quad N_2 = N_h + N_\downarrow \]

Nested Bethe ansatz equations:

\[
\left( \frac{\lambda^{(2)}_p + \frac{i}{2}}{\lambda^{(2)}_p - \frac{i}{2}} \right)^L = \prod_{\substack{q=1 \\ q \neq p}}^{N_h+N_\downarrow} \frac{\lambda^{(2)}_p - \lambda^{(2)}_q + i}{\lambda^{(2)}_p - \lambda^{(2)}_q - i} \prod_{i=1}^{N_h} \frac{\lambda^{(2)}_p - \lambda^{(1)}_i - \frac{i}{2}}{\lambda^{(2)}_p - \lambda^{(1)}_i + \frac{i}{2}}, \quad p = 1, \ldots, N_h + N_\downarrow, \\
1 = \prod_{p=1}^{N_h+N_\downarrow} \frac{\lambda^{(2)}_p - \lambda^{(1)}_i - \frac{i}{2}}{\lambda^{(2)}_p - \lambda^{(1)}_i + \frac{i}{2}}, \quad i = 1, \ldots, N_h.
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Yang Yang counting function:

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Y_A(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \hat{x}(2\lambda^{(2)}_p) - \frac{1}{2\pi} \sum_{p,q=1}^{N_h+N_\downarrow} \hat{x}(\lambda^{(2)}_p - \lambda^{(2)}_q) + \frac{1}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \sum_{i=1}^{N_h} \hat{x}(2\lambda^{(2)}_p - 2\lambda^{(1)}_i) \\
\hat{x}(\lambda) = \lambda \arctan(\lambda^{-1}) + \frac{1}{2} \log(1 + \lambda^2) \\
+ \sum_{i=1}^{N_h} \lambda^{(1)}_i + \sum_{p=1}^{N_h+N_\downarrow} n^{(2)}_p \lambda^{(2)}_p.
\]
Bethe ansatz

Case B:

\[ C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda = [0 \ 1], \quad N_1 = N_\downarrow, \quad N_2 = N_h + N_\downarrow. \]

Nested Bethe ansatz equations:

\[
\left( \frac{\lambda_p^{(2)} + i/2}{\lambda_p^{(2)} - i/2} \right)^L = \prod_{i=1}^{N_\downarrow} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - i/2}{\lambda_i^{(1)} - \lambda_p^{(2)} + i/2}, \quad p = 1, \ldots, N_h + N_\downarrow
\]

\[
1 = \prod_{p=1}^{N_h+N_\downarrow} \frac{\lambda_i^{(1)} - \lambda_p^{(2)} - i/2}{\lambda_i^{(1)} - \lambda_p^{(2)} + i/2}, \quad i = 1, \ldots, N_\downarrow.
\]

Yang Yang counting function:

\[
Y_B(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \hat{x}(2\lambda_p^{(2)}) + \frac{1}{2\pi} \sum_{p=1}^{N_h+N_\downarrow} \sum_{i=1}^{N_1} \hat{x}(2\lambda_i^{(1)} - 2\lambda_p^{(2)}) + \sum_{i=1}^{N_1} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_h+N_\downarrow} n_p^{(2)} \lambda_p^{(2)}
\]
Bethe ansatz

Case C:

\[ C^{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda = [0 \ 1], \quad N_1 = N_\downarrow, \quad N_2 = N_\uparrow + N_\downarrow. \]

Nested Bethe ansatz equations:

\[
\left( \frac{\lambda_p^{(2)} - \frac{i}{2}}{\lambda_p^{(2)} + \frac{i}{2}} \right)^L = \prod_{i=1}^{N_\downarrow} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}}, \quad p = 1, \ldots, N_e,
\]

\[
\prod_{p=1}^{N_e} \frac{\lambda_p^{(2)} - \lambda_i^{(1)} - \frac{i}{2}}{\lambda_p^{(2)} - \lambda_i^{(1)} + \frac{i}{2}} = \prod_{j=1 \atop j \neq i}^{N_\downarrow} \frac{\lambda_j^{(1)} - \lambda_i^{(1)} - i}{\lambda_j^{(1)} - \lambda_i^{(1)} + i}, \quad i = 1, \ldots, N_\downarrow.
\]

Yang Yang counting function:

\[
Y_C(\lambda) = \frac{L}{2\pi} \sum_{p=1}^{N_e} \hat{x}(2\lambda_p^{(2)}) - \frac{1}{2\pi} \sum_{p=1}^{N_e} \sum_{i=1}^{N_\downarrow} \hat{x}(2\lambda_p^{(2)} - 2\lambda_i^{(1)}) + \frac{1}{2\pi} \sum_{i,j=1 \atop i \neq j}^{N_\downarrow} \hat{x}(\lambda_i^{(1)} - \lambda_j^{(1)})
\]

\[
+ \sum_{i=1}^{N_\downarrow} n_i^{(1)} \lambda_i^{(1)} + \sum_{p=1}^{N_e} n_p^{(2)} \lambda_p^{(2)}.
\]
Dictionary
The Dictionary

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The Dictionary

The tJ model:
- spin chain of length L
- periodic boundary conditions
- 3 particle species: spin up, spin down, empty
- \(sl(1|2)\) symmetry
- most important feature: 3 inequivalent choices of Cartan matrix
- 3 different (but equivalent) sets of Bethe equations
- spectrum the same for all 3 cases
- fundamental rep. (spin 1/2) at each lattice point
- no inhomogeneities

\(U(L)\) flavor group

two nodes in quiver

masses for adj. and bifund. fields

\(3\) gauge theories

same susy ground states!
The Dictionary

Use Gauge/Bethe dictionary:

<table>
<thead>
<tr>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Diagram A" /></td>
<td><img src="image2" alt="Diagram B" /></td>
<td><img src="image3" alt="Diagram C" /></td>
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</tbody>
</table>

These 3 quiver gauge theories have the same ground states!

Conversely: any quiver gauge theories which can be associated to the same int. model have same ground states.
**Quiver Gauge Theory**

Case A: \( C^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \), \( \Lambda = [0 \ 1] \), \( N_1 = N_h \), \( N_2 = N_h + N_\perp \)

- **gauge groups**
  - \( U(N_h) \)
  - \( U(N_h + N_\perp) \)

- **bifundamental fields**
  - \( B^{12} \)
  - \( B^{21} \)
  - \( N_h \otimes (N_h + N_\perp) \)
  - \( (N_h + N_\perp) \otimes N_h \)

- **adjoint field** \( \Phi^2 \)

- **fundamental and antifundamental fields** \( (Q^2_k, \bar{Q}^2_k) \)

- **flavor group**
  - \( \text{U}(L) \)
  - \( \text{U}(L) \)

- **twisted masses**
  - \( \frac{1}{2} \)
  - \( -\frac{1}{2} \)

**Global symmetry group:**

\[ U(L)_Q \times U(L)_{\tilde{Q}} \times U(1)_B \times U(1)_{\tilde{B}} \times U(1)_\Phi \]

**Broken down to maximal torus by twisted masses.**
Quiver Gauge Theory

Effective twisted superpotential:

$$\tilde{W}_{\text{eff}}^A(\sigma) = \frac{L}{2\pi} \sum_{p=1}^{N_h+N_\perp} \left[ \left( \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(\sigma_p^{(2)} + \frac{i}{2}) - 1 \right) - \left( \sigma_p^{(2)} - \frac{i}{2} \right) \left( \log(-\sigma_p^{(2)} + \frac{i}{2}) - 1 \right) \right]$$

$$+ \frac{1}{2\pi} \sum_{i=1}^{N_h} \sum_{p=1}^{N_h+N_\perp} \left[ \left( \sigma_i^{(1)} - \sigma_p^{(2)} - \frac{i}{2} \right) \left( \log(\sigma_i^{(1)} - \sigma_p^{(2)} - \frac{i}{2}) - 1 \right) \right]$$

$$- \left( \sigma_i^{(1)} - \sigma_p^{(2)} + \frac{i}{2} \right) \left( \log(-\sigma_i^{(1)} + \sigma_p^{(2)} - \frac{i}{2}) - 1 \right)$$

$$+ \frac{1}{2\pi} \sum_{p,q}^{N_h+N_\perp} \left( \sigma_p^{(2)} - \sigma_q^{(2)} - \nu \right) \left( \log(\sigma_p^{(2)} - \sigma_q^{(2)} - \nu) - 1 \right) - \nu_1 \sum_{i=1}^{N_h} \sigma_i^{(1)} - \nu_2 \sum_{p=1}^{N_h+N_\perp} \sigma_p^{(2)}.$$  

Corresponds to $Y$.

Superpotential (compatible to eff. tw. superpotential):

$$W_A(Q^2, \overline{Q}^2, \Phi^2, B^{12}, \overline{B}^{21}) = \sum_k \left[ a Q^2_k \Phi^2 \overline{Q}^2_k + b Q^2_k B^{21} B^{12} \overline{Q}^2_k \right]$$
Relation via Brane Cartoons

Different argument for relation: brane motions

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Hanany, Witten; Hanany, Hori

Not surprising that the relation between the 3 theories can be seen in different ways.

But: difficult to turn on twisted masses!
The Big Picture
The Big Picture

For the integrable system, all $N$ magnon sectors must be considered together.

This is usually not done for $U(N)$ gauge theories with different $N$.

Yet, there is evidence that it would make sense to do so.

Study the simplest example of the spin 1/2 XXX spin chain (Heisenberg model).

The low energy limit of the corresponding gauge theory is given by the NLSM with target space the cotangent bundle of $\text{Gr}(N,L)$.

$$\text{Gr}(N,L) = \{ W \subset \mathbb{C}^L | \dim W = N \},$$

$$T^*\text{Gr}(N,L) = \{ (X, W), W \in \text{Gr}(N,L), X \in \text{End}(\mathbb{C}^L) | X(\mathbb{C}^L) \subset W, X(W) = 0 \}$$
The Big Picture

Its ground states are given by the cohomology of $T^* \text{Gr}(N,L)$

A representation of $\text{su}(2)$ is given by a vector space $V$ and the operators $e,f,k$:

$$[e, f] = k, \quad [k, e] = 2e, \quad [k, f] = -2f.$$  

Geometric representation theory provides the following construction:

$$V \otimes L = \bigoplus_{N=0}^{L} V_{L-2N} \cong \bigoplus_{N=0}^{L} H_\ast[T^* \text{Gr}(N, L)]$$

**Construct Hecke operators $e, f$:**

$$f : H_\ast[T^* \text{Gr}(N, L)] \to H_\ast[T^* \text{Gr}(N + 1, L)]$$

Hilbert space of spin chain  
weight space  
ground states of gauge theory
The Big Picture

Introduce a correspondence:

$$Z \subset T^* \text{Gr}(N, L) \times T^* \text{Gr}(N + 1, L)$$

$$x \mapsto f(x) = \pi_2^*([Z] \cap \pi_1^*(x))$$

$$e : H_*[T^* \text{Gr}(N + 1, L)] \to H_*[T^* \text{Gr}(N, L)]$$

$$x \mapsto e(x) = (-1)^L \pi_1^*([Z] \cap \pi_2^*(x))$$

$$k^* = \bigoplus_{N=0}^{L} (L - 2N) \text{Id}_{H_*[T^* \text{Gr}(N, L)]}$$
The Big Picture

Math/phys dictionary:

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<th>physics</th>
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<tr>
<td>spectrum of $\sigma_{1/2}$ spin chain</td>
<td>$\text{su}(2)$ representation $V \otimes L \simeq H_<em>[T^</em>\text{Gr}(L)]$</td>
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<tr>
<td>ground states of the NLSM on $T^*\text{Gr}(N,L)$</td>
<td>cohomology $H_<em>[T^</em>\text{Gr}(N,L)]$</td>
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<tr>
<td>spectrum for the $N$ magnon sector</td>
<td>weight space $V_{L-2N} \simeq H_<em>[T^</em>\text{Gr}(N,L)]$</td>
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<tr>
<td>ground states of $\sigma_{1/2}$</td>
<td>hw representation $V(L) \simeq H_{\text{top}}[T^*\text{Gr}(L)]$</td>
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<tr>
<td>gauge/Bethe correspondence</td>
<td>geometric representation of $\text{su}_2$</td>
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**Diagram:**

- $H_*[T^*\text{Gr}(1,4)] \simeq V_2$
- Symmetry: $N, L - N$
- $\text{Gr}(1,4), \text{Gr}(3,4)$
- Irreps of $\text{su}(2)$
Summary
Summary

The Gauge/Bethe correspondence of Nekrasov/Shatashvili relates the susy ground states of a 2d N=(2,2) U(N) gauge theory to the N magnon spectrum of an integrable model.

We use this correspondence as a tool to relate different quiver gauge theories which correspond to the same integrable system.

Works for all integrable models with supergroup symmetry.

Window from the very well controlled integrable models into gauge theory.
Summary

Any spin chain with supergroup symmetry gives rise to several quiver gauge theories.

The supergroup $sl(m|n)$ gives rise to $\binom{m+n}{m}$ distinct quiver gauge theories.

Open questions:

Understand the meaning of twisted masses in gauge theory better.

Reproduce twisted masses in brane realizations.

Study and compare soliton solutions for the different quiver gauge theories corresponding to one spin chain.
Summary

Geometric representation theory supplies the mathematical framework which underlies the Gauge/Bethe correspondence.

The case of the su(2) Heisenberg spin chain corresponds directly to the construction of Ginzburg for the cotangent bundle of the Grassmannian, while the su(n) spin chain corresponds to the construction for flag manifolds.

One can define the Hecke operators $e$ and $f$ which pass from $N$ to $N+1$ and vice versa.
Summary

$\text{su}(2)$ symmetry and integrable structure of the spin chain are only manifest in this setting.

Further directions:

Generalize this construction to all Lie groups/supergroups.

Generalize to the equivariant case.

The Yangian is the most important structure in the integrable model. Generalize to this case.

The operators $e$ and $f$ could be interpreted as brane creation and annihilation operators.
Thank you for your attention!